Homework 1

Ma121a Combinatorial Analysis
Due Thursday October 11, 2007, 5pm .

Q1. (Problem 4A) Let G be a simple graph with 10 vertices and 26 edges. Show that G has at least 5 triangles. Can equality occur?

Q2. Let $ex(n, F)$ denote the maximum number of edges in a graph on n vertices not containing a copy of F as a subgraph. For $0 < s \leq t \leq n$ let $z(n, s, t)$ denote the maximum number of edges in a bipartite graph whose partition sets both have size n, and which does not contain a $K_{s,t}$. Show that

$$2 \cdot ex(n, K_{s,t}) \leq z(n, s, t) \leq ex(2n, K_{s,t}).$$

Q3. Let $1 \leq r \leq n$ be integers. Let G be a bipartite graph with bipartition $\{A, B\}$, where $|A| = |B| = n$, and assume that $K_{r,r} \not\subseteq G$. Show that

$$\sum_{x \in A} \binom{d(x)}{r} \leq (r-1)\binom{n}{r}.$$

Using the previous exercise, deduce that $ex(n, K_{r,r}) \leq cn^{2-\frac{1}{r}}$ for some constant c depending only on r.

Hint: you may want to use the estimation $(\frac{s}{t})^t \leq \binom{s}{t} \leq s^t$ and the **Jensen's Inequality:** Let f be a convex function and $w_1, \ldots, w_n$ non-negative reals such that $\sum w_i = 1$. Then

$$\sum_{i=1}^{n} w_i f(x_i) \geq f\left(\sum_{i=1}^{n} w_i x_i\right).$$

Q4. The *upper density* of an infinite graph G is the infimum of all reals α such that the finite graphs $H \subseteq G$ with $\|H\|\binom{|H|}{2}^{-1} > \alpha$ have bounded order ($\|H\|$ and $|H|$ denote the number of edges and the order of H respectively). Use the Erdős-Stone Theorem to show that this number always takes one of the countably many values $0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$.

Q5. (Problem 4C) Prove that if a simple graph on n vertices has e edges, then it has at least $\frac{e}{3n}(4e - n^2)$ triangles.

Q6. (Problem 4H) Show that a graph on n vertices that does not contain a circuit on four vertices has at most $\frac{n}{4}(1 + \sqrt{4n-3})$ edges.