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ACM 104: HW 2

Exercise 1. Let V and W be finite-dimentsional vector spaces and let $T : V \to W$ be a linear transformation. Let β be an ordered basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is an ordered basis for W.

Proof. Assume $\dim(V) = n$, T is an isomorphism and there are α_i such that

$$\sum_{i=1}^{n} \alpha_i T\beta_i = 0 \Rightarrow \alpha_1 T\beta_1 = -\sum_{i=2}^{n} \alpha_i T\beta_i \Rightarrow T(\alpha_1\beta_1) = T\left(-\sum_{i=2}^{n} \alpha_i \beta_i\right).$$

Since T is one-to-one, this implies

$$\alpha_1\beta_1 = -\sum_{i=2}^{n} \alpha_i\beta_i \Rightarrow \sum_{i=1}^{n} \alpha_i\beta_i = 0$$

and since β is linearly independent, $\alpha_i = 0$ for all i, showing $T(\beta)$ is also linearly independent. Therefore T is an isomorphism implies $T(\beta)$ is a basis.

Assume $T(\beta)$ is a basis, then clearly T is onto. Furthermore, let $x \in \mathcal{N}(T)$, then there are α_i such that

$$x = \sum_{i=1}^{n} \alpha_i\beta_i \Rightarrow 0 = Tx = \sum_{i=1}^{n} \alpha_i T\beta_i,$$

and since $T(\beta)$ is linearly independent, this implies $\alpha_i = 0$ for all i. Therefore $x = 0$, so $\mathcal{N}(T) = \{0\}$ and T is one-to-one. Therefore $T(\beta)$ a basis implies T is an isomorphism. ☐

Exercise 2. Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Let $Q = [I]_{\beta'}^{\beta}$ and $P = [I]_{\gamma'}^{\gamma}$. Prove that

$$[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q.$$

Proof. Using Theorem 2.16 twice,

$$[T]_{\beta'}^{\gamma'} = [IT]_{\beta'}^{\gamma'} = [I]_{\gamma}^{\gamma'}[T]_{\beta'}^{\gamma} = [I]_{\gamma}^{\gamma'}[TI]_{\beta'}^{\gamma} = [I]_{\gamma}^{\gamma'}[T]_{\beta}^{\gamma}[I]_{\beta'}^{\beta}.$$

By Theorem 2.21, $[I]_{\gamma}^{\gamma'} = [I^{-1}]_{\gamma}^{\gamma'} = \left([I]_{\gamma'}^{\gamma}\right)^{-1}$, so

$$[T]_{\beta'}^{\gamma'} = \left([I]_{\gamma'}^{\gamma}\right)^{-1}[T]_{\beta}^{\gamma}[I]_{\beta'}^{\beta} = P^{-1}[I]_{\beta}^{\gamma}Q.$$

☐

Exercise 3. Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows:

$$f_1(x, y, z) = x - 2y, \quad f_2(x, y, z) = x + y + z, \quad f_3(x, y, z) = y - 3z.$$

Prove that $\beta^* = \{f_1, f_2, f_3\}$ is a basis for V^*. Moreover, find a basis β for V such that the dual basis of β is β^*.

Proof. Let $\alpha_1, \alpha_2, \alpha_3$ be such that $\sum \alpha_i f_i = 0$, i.e. for all $(x, y, z) \in V$,

$$\alpha_1(x - 2y) + \alpha_2(x + y + z) + \alpha_3(y - 3z) = (\alpha_1 + \alpha_2)x + (-2\alpha_1 + \alpha_2 + \alpha_3)y + (\alpha_2 - 3\alpha_3)z = 0,$$

and choosing the values $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ for (x,y,z) gives

$$\begin{aligned}\alpha_1+\alpha_2&=0\\ -2\alpha_1+\alpha_2+\alpha_3&=0\\ \alpha_2-\alpha_3&=0\end{aligned}$$

or $\alpha_2=-\alpha_1$ and $\alpha_3=\frac{1}{3}\alpha_2=-\frac{1}{3}\alpha_1$, so $-2\alpha_1-\alpha_1-\frac{1}{3}\alpha_1=0\Rightarrow\alpha_1=0$, so $\alpha_1=\alpha_2=\alpha_3=0$ and f_1,f_2,f_3 are linearly independent. Since $\dim(V^*)=\dim(V)=3$, β is a basis for V^*.

Since $E^*=\{x,y,z\}$ is the dual basis to $E=\{e_1,e_2,e_3\}$, if $v=v_1e_1+v_2e_2+v_3e_3\in V$ and $f=w_1x+w_2y+w_3z\in V^*$, then

$$f(v)=w_1x(v)+w_2y(v)+w_3z(v)=w_1v_1+w_2v_2+w_3v_3=[f]_{E^*}^t[v]_E.$$

If $\beta=\{\beta_1,\beta_2,\beta_3\}$ is such that $\beta^*=\{f_1,f_2,f_3\}$, then

$$f_i(B_j)=\delta_{i,j}=[f_i]_{E^*}^t[B_j]_E,$$

or

$$\begin{pmatrix}1&-2&0\\1&1&1\\0&1&-3\end{pmatrix}\begin{pmatrix}b_{1,1}&b_{1,2}&b_{1,3}\\b_{2,1}&b_{2,2}&b_{2,3}\\b_{3,1}&b_{3,2}&b_{3,3}\end{pmatrix}=I,$$

where $B_j=\sum_{i=1}^3 b_{i,j}e_i$. Note that this condition both necessary and sufficient: by the uniqueness of the inverse of the matrix, this relationship is satisfied by exactly one set of $\{b_{i,j}\}$, so the β which has these $\{b_{i,j}\}$ as coefficients satisfies $\beta^*=\{f_1,f_2,f_3\}$.

Since

$$\begin{pmatrix}b_{1,1}&b_{1,2}&b_{1,3}\\b_{2,1}&b_{2,2}&b_{2,3}\\b_{3,1}&b_{3,2}&b_{3,3}\end{pmatrix}=\begin{pmatrix}1&-2&0\\1&1&1\\0&1&-3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{2}{5}&\frac{3}{5}&\frac{1}{5}\\\frac{-3}{10}&\frac{3}{10}&\frac{1}{10}\\\frac{-1}{10}&\frac{1}{10}&\frac{-3}{10}\end{pmatrix},$$

$$\beta=\left\{\frac{2}{5}e_1-\frac{3}{10}e_2-\frac{1}{10}e_3,\frac{3}{5}e_1+\frac{3}{10}e_2+\frac{1}{10}e_3,-\frac{1}{10}e_1+\frac{1}{10}e_2-\frac{3}{10}e_3\right\}$$

is such that $\{f_1,f_2,f_3\}$ is the dual basis to β. ☐

Exercise 4. Let V be a finite-dimensional vector space with ordered basis β. Let ψ be the isomorphism between V and V^{**} defined by $\psi(x)=\hat{x}$. Prove that $\phi(\beta)=\beta^{**}$.

Proof. Let $\beta=\{\beta_1,\dots,\beta_n\}$ and $\beta^*=\{\beta_1^*,\dots,\beta_n^*\}$, then $\psi(\beta)=\{\hat{\beta}_1,\dots,\hat{\beta}_n\}\subset V^{**}$ and $\hat{\beta}_i(\beta_j^*)=\beta_j^*(\beta_i)=\delta_{ij}$. In particular, if $v=\sum_{j=1}^n\alpha_j\beta_j^*\in V*$,

$$\hat{\beta}_i(v)=\hat{\beta}_i\left(\sum_{j=1}^n\alpha_j\beta_j^*\right)=\sum_{j=1}^n\alpha_j\delta_{ij}=\alpha_i,$$

so $\hat{\beta}_i$ is the i-th coordinate function with respect to the basis β^*. Therefore by definition $\psi(\beta)$ is the dual basis of β^*, or $\psi(\beta)=\beta^{**}$. ☐

Exercise 5. Let V and W be finite-dimensional vector spaces and let $T:V\to W$ be a linear transformation. Let ψ_1 be the isomorphism between V and V^* defined by $\psi_1(x)=\hat{x}$ and ψ_2 be the isomorphism between W and W^{**} defined by $\psi_2(y)=\hat{y}$. Show that

$$\psi_2T=T^{tt}\psi_1.$$

Proof. Let $x \in V$ and $w^* \in W^*$, then $T^{tt}\psi_1 : V \to W^{**}$ and $\psi_2 T : V \to W^{**}$, and

$$(T^{tt}\psi_1 x)(w^*) = (T^{tt}\hat{x})(w^*) = \hat{x}(T^t w^*) = (T^t w^*)(x) = w^*(Tx) = \widehat{Tx}(w^*) = (\psi_2 Tx)(w^*),$$

so $T^{tt}\psi_1 = \psi_2 T$. $\square$

Exercise 6. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f) = f(x) + xf'(x)$. Find the eigenvalues of T, and find an ordered basis for $P_2(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

Answer. Let $\beta = \{1, x, x^2\}$, then $T(1) = 1$, $T(x) = 2x$, and $T(x^2) = 3x^2$, so by definition

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

so β is a basis in which $[T]_\beta$ is diagonal. Furthermore,

$$\det(T - \lambda I) = \det([T]_\beta - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda),$$

so the eigenvalues of T are $1, 2, 3$.

Exercise 7. A **scalar matrix** is a square matrix of the form λI for some scalar λ.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.
(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.
(c) Show that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable.

Proof. (a) If $A \sim \lambda I$, there is an invertible matrix Q such that $A = Q^{-1}\lambda I Q = \lambda Q^{-1} I Q = \lambda Q^{-1}Q = \lambda I$, so $A = \lambda I$.

(b) If D is a diagonalizable matrix with one eigenvalue λ, then there is an invertible matrix Q such that

$$D = Q^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} Q.$$

By (a) above, since D is similar to a scalar matrix, it is a scalar matrix.

(c) Let A be the given matrix, then $\det(A - \lambda I) = (1-\lambda)^2$, so the eigenvalues of A are both 1. By (b) above, if A were diagonalizable, it would be a scalar matrix; since A is not a scalar matrix, it is not diagonalizable.

$\square$

Exercise 8. Let $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be a linear operator defined by $T(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.
(b) For $n = 2$, find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is diagonal.
(c) For $n > 2$, find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is diagonal.

Answer.

(a) Assume λ is an eigenvalue of T, then there is a non-zero $A \in M_{n\times n}(\mathbb{R})$ such that $T(A) = \lambda A$, so

$$A = \left(A^t\right)^t = T^2(A) = T(\lambda A) = \lambda^2 A,$$

so $1 = \lambda^2 \Rightarrow \lambda \in \{1, -1\}$, so the only potential eigenvalues of T are ± 1. But clearly I is an eigenvector corresponding to eigenvalue 1, and the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & \dots \\ -1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is an eigenvector corresponding to eigenvalue -1, so 1 and -1 are the only eigenvalues of T.

(b) From the argument in (c) below, the ordered basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

is such that $[T]_\beta$ is diagonal.

(c) Let $n \geq 2$, then to diagonalize T it is necessary to find a basis of eigenvectors, that is, of symmetric and skew-symmetric matrices. Let $S_+^n \subset M_{n\times n}$ and $S_-^n \subset M_{n\times n}$ denote respectively the subspace of symmetric matrices and the subspace of skew-symmetric matrices. Note that $-A^t = A = A^t \Rightarrow A = \{0\}$, so $S_+^n \cap S_-^n = \{0\}$; also, if $A \in M_{n\times n}$,

$$A = \left(\frac{A + A^t}{2}\right) + \left(\frac{A - A^t}{2}\right),$$

where the first matrix in the decomposition is in S_+^n and the second is in S_-^n. This shows $M_{n\times n} = S_+^n \oplus S_-^n$, so if β_+ is an ordered basis for S_+^n and β_- is an ordered basis for S_-^n, then $\beta = \beta_+ \cup \beta_-$ is an ordered basis for $M_{n\times n}$.

Note that every symmetric $n\times n$ matrix can be written as a unique linear combination of the $n(n+1)/2$ matrices $U_{i,j}$ where $i, j \in \{1, \dots, n\}$, $i \leq j$, and $(U_{i,j})_{r,c} = \delta_{i,r}\delta_{j,c} + \delta_{i,c}\delta_{j,r}$; this shows the set $\beta_+ = \{E_{i,j} : i, j \in \{1, \dots, n\} \text{ and } i \leq j\}$ is a basis for S_+^n. Likewise every skew-symmetric $n \times n$ matrix can be written as a unique linear combination of the $n(n-1)/2$ matrices $L_{i,j}$ where $i, j \in \{1, \dots, n\}$, $j < i$, and $(L_{i,j})_{r,c} = \delta_{i,r}\delta_{j,c} - \delta_{i,c}\delta_{j,r}$; this shows the set $\beta_- = \{L_{i,j} : i, j \in \{1, \dots, n\} \text{ and } j < i\}$ is a basis for S_-^n.

Therefore $\beta = \beta_+ \cup \beta_-$ is a basis of $M_{n\times n}$ consisting of eigenvectors of T, so $[T]_\beta$ is diagonal.