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ACM 104: HW 6

Exercise 1. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following:

(a) If T is self-adjoint, then T_W is self-adjoint.
(b) $W^\perp$ is T^*-invariant.
(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

Proof. (a) Let $x \in W$, then

$$\langle T_W x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, T_W x\rangle,$$

so T_W is self-adjoint.

(b) Let $y \in W^\perp$ and $x \in W$ then

$$\langle T^* y, x\rangle = \langle y, Tx\rangle = 0$$

since $Tx \in W$, so $T^* y \in W^\perp$. That is, $W^\perp$ is T^*-invariant.

(c) Assume W is both T- and T^* invariant. Let $x, y \in W$, then

$$\langle T_W x, y\rangle = \langle Tx, y\rangle = \langle x, T^* y\rangle = \langle x, (T^*)_W y\rangle,$$

so $(T_W)^* = (T^*)_W$. Define $T_W^* = (T^*)_W = (T_W)^*$.

(d) Assume W is both T- and T^*-invariant and T is normal. Let $x \in W$, then

$$T_W T_W^* x = T_W T^* x = TT^* x = T^* T x = T^* T_W x = T_W^* T_W x,$$

where each step is well-defined because of the T- and T^*-invariance of W. Since T_W and T_W^* commute, T_W is normal.

□

Exercise 2. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$.

Proof. Notice

$$x \in \mathcal{N}(T) \Leftrightarrow 0 = \|Tx\| = \|T^* x\| \Leftrightarrow x \in \mathcal{N}(T^*),$$

so $\mathcal{N}(T) = \mathcal{N}(T^*)$.

Furthermore,

$$y \in \mathcal{R}(T)^\perp \Leftrightarrow \langle T(T^* y), y\rangle = 0 \Leftrightarrow \langle T^* y, T^* y\rangle = 0 \Leftrightarrow T^* y = 0 \Leftrightarrow y \in \mathcal{N}(T^*),$$

so $\mathcal{R}(T)^\perp = \mathcal{N}(T^*)$; similarly, $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$. Consequently, $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp = \mathcal{N}(T)^\perp = \mathcal{R}(T^*)$. □

Exercise 3. Let T be a normal operator on a finite-dimensional real inner product space V such that the characteristic polynomial of T splits. Prove that V has an orthonormal basis of eigenvectors of T.

Adaptation of Thm 5.29 in the class notes. Since the characteristic polynomial of T splits, by Schur's theorem, there is an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$ for V such that $A = [T]_\beta$ is upper triangular. In particular, v_1 is an eigenvector of T. Assume $v_1, \dots, v_{k-1}$ are eigenvectors of T. Write

$$A = \begin{pmatrix} B & C \\ 0 & E \end{pmatrix}, \quad A^* = \begin{pmatrix} B^* & 0 \\ C^* & E^* \end{pmatrix}.$$

Since $v_1, \dots, v_{k-1}$ are eigenvectors of T, B is a $(k-1) \times (k-1)$ diagonal matrix. Moreover, we have $A_{jk} = 0$ if $j > k$ since A is upper triangular. Since T is normal, $v_1, \dots, v_{k-1}$ are also eigenvectors of T^*; since $A^* = [T^*]_\beta$, this means $C^* = 0$. Since E is upper triangular and $C = 0$, the only nonzero entry in the k-th column of A is A_{kk}, so v_k is an eigenvector of A.

Therefore $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis for V consisting of eigenvectors of T. ☐

Exercise 4. Let V be a finite-dimensional real inner product space. Let U and T be self-adjoint operators such that $UT = TU$. Prove that there exists an orthonormal basis for V consisting of eigenvectors of both U and T. (Hint: Let $W = E_\lambda$ be an eigenspace of T. Prove that W is both T- and U-invariant.)

Proof. Given $w \in E_{\lambda_i}$,

$$TUw = UTw = \lambda_i Uw \Rightarrow Uw \in E_{\lambda_i},$$

so E_{λ_i} is U-invariant. By (1a) above, $U_{E_{\lambda_i}}$ is self-adjoint, so there is an orthonormal basis β_i for E_{λ_i} consisting of eigenvectors of U which are also eigenvectors of T. Since T is self-adjoint, it is diagonalizable, so $V = \oplus_i E_{\lambda_i}$. Therefore $\beta = \cup_i \beta_i$ is an orthonormal basis for V consisting of eigenvectors of both U and T. ☐

Exercise 5. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i, \quad \operatorname{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2,$$

where λ_i are the eigenvalues (not necessarily distinct) of A.

Proof. Let A be a $n \times n$ real symmetric (therefore self-adjoint) or complex normal matrix. Then A is unitarily similar to a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A; that is, there is a unitary matrix Q such that $A = Q^{-1}DQ$. Then, using Einstein notation:

$$\begin{aligned} \operatorname{tr}(A) = A_{ii} = (Q^{-1}DQ)_{ii} &= Q^{-1}_{ij}(DQ)_{ji} = Q^{-1}_{ij} D_{jr} Q_{ri} \\ &= Q_{ri} Q^{-1}_{ij} D_{jr} = (QQ^{-1})_{rj} D_{jr} = \delta_{rj} D_{jr} = D_{rr} = \operatorname{tr}(D) = \sum_i \lambda_i. \end{aligned}$$

Since $Ax = \lambda x \Leftrightarrow A^*x = \overline{\lambda} x \Rightarrow A^*Ax = \overline{\lambda}\lambda x = |\lambda|^2 x$, A^*A shares the eigenvectors of A, so $A^*A = Q^{-1}D'Q$ where $D' = \operatorname{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2)$, so as above

$$\operatorname{tr}(A^*A) = \operatorname{tr}(D') = \sum_i |\lambda_i|^2.$$

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Exercise 6. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove

(a) $U(W) = W$.
(b) $W^{\perp}$ is U-invariant.

Proof. (a) Since for all $x \in W$

$$\|x\| = \|Ux\| = \|U_W x\|,$$

U_W is unitary. Therefore there is an orthonormal basis β of W such that $U(\beta)$ is an orthonormal basis for W. Let $y \in W$, $y = \sum \alpha_i U(\beta_i)$, then $y = U(\sum \alpha_i \beta_i)$ where $\sum \alpha_i \beta_i \in W$, so $y \in U(W)$; that is, $W \subset U(W)$. But W is U-invariant, so $U(W) \subset W$. Therefore $W = U(W)$.

(b) Let $z \in W^{\perp}$, $x \in W$. Since $W = U(W)$, $U^* x = U^{-1} x \in W$, so

$$\langle Uz, x\rangle = \langle z, U^* x\rangle = 0,$$

and $Uz \in W^{\perp}$. Therefore $W^{\perp}$ is U-invariant.

□

Exercise 7. Prove that a matrix that is both unitary and upper triangular must be a diagonal matrix.

Proof. Let A and B be upper triangular matrices, then

$$(AB)_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj} = \sum_{r=i}^{n} A_{ir} B_{rj}.$$

Assume $i > j$, then in this sum $r > j$ always, so $B_{rj} = 0$ always, and $(AB)_{ij} = 0$. Therefore the set of upper triangular matrices is a subspace of $\mathbb{C}^{n \times n}$ closed under multiplication.

Let U be an upper triangular unitary matrix, then by exercise 4 in pset 3, $U^* = U^{-1} = p(U)$ for some polynomial p, so U^* is upper triangular. Yet U is upper triangular, so U^* is lower triangular. Therefore U^* and consequently also U are diagonal. □

Exercise 8. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

Deduce that $(T - iI)$ is invertible and that $((T - iI)^{-1})^* = (T + iI)^{-1}$.

Proof. Let $x \in V$, then

$$\begin{aligned}
\|Tx \pm ix\|^2 &= \langle Tx \pm ix, Tx \pm ix\rangle = \langle Tx, Tx\rangle \pm \langle ix, Tx\rangle \pm \langle Tx, ix\rangle + \langle ix, ix\rangle \\
&= \|Tx\|^2 + i\bar{i}\|x\|^2 \pm i\langle x, Tx\rangle \mp i\langle Tx, x\rangle \\
&= \|Tx\|^2 + \|x\|^2 \pm i\langle x, Tx\rangle \mp \langle x, Tx\rangle \\
&= \|Tx\|^2 + \|x\|^2.
\end{aligned}$$

In particular,

$$x \in \mathcal{N}(T - iI) \Leftrightarrow \|Tx - ix\| = 0 \Leftrightarrow \|Tx\|^2 + \|x\|^2 = 0 \Leftrightarrow x = 0,$$

so $T - iI$ is invertible.

Furthermore, since T is self-adjoint, $(T - iI)^* = T + iI$. By exercise 7 in problem set 5,

$$(T + iI)^{-1} = ((T - iI)^{-1})^*.$$

□

Exercise 9. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that $(T+iI)(T-iI)^{-1}$ is unitary.

Proof. First note

$$(T-iI)(T+iI) = T^2 + I = (T+iI)(T-iI),$$

so $T+iI$ and $T-iI$ commute.

Then

$$\left((T+iI)(T-iI)^{-1}\right)^* = \left((T-iI)^{-1}\right)^* (T+iI)^* = (T+iI)^{-1}(T-iI),$$

so

$$\begin{aligned}\left((T+iI)(T-iI)^{-1}\right)\left((T+iI)(T-iI)^{-1}\right)^* &= (T+iI)(T-iI)^{-1}(T+iI)^{-1}(T-iI)\\ &= (T+iI)\left((T+iI)(T-iI)\right)^{-1}(T-iI)\\ &= (T+iI)\left((T-iI)(T+iI)\right)^{-1}(T-iI)\\ &= (T+iI)(T+iI)^{-1}(T-iI)^{-1}(T-iI)\\ &= I,\end{aligned}$$

showing $(T+iI)(T-iI)^{-1}$ is unitary. □

Exercise 10. Let V be a finite-dimensional complex inner product space and let u be a unit vector in V. Define the Householder operator $H_u : V \to V$ by

$$H_u(x) = x - 2\langle x, u\rangle u.$$

Prove that

(1) $H_u^* = H_u$.
(2) $H_u^2 = I$.

Hence H_u is unitary.

Proof. (a) Let $x, y \in V$, then

$$\begin{aligned}\langle H_u x, y\rangle = \langle x - 2\langle x,u\rangle u, y\rangle = \langle x, y\rangle - 2\langle x, u\rangle\langle u, y\rangle = \langle x, y\rangle - \langle x, 2\langle y, u\rangle u\rangle\\ = \langle x, y - 2\langle y, u\rangle u\rangle = \langle x, H_u y\rangle,\end{aligned}$$

so H_u is self-adjoint.

(b) Let $x \in V$, then

$$\begin{aligned}H_u^2 x &= H_u x - 2\langle H_u x, u\rangle u = H_u x - 2\langle x, H_u u\rangle u = x - 2\langle x, u\rangle u - 2\langle x, u - 2u\|u\|^2\rangle u\\ &= x - 2\langle x, u\rangle u - 2\langle x, u\rangle u + 4\langle x, u\rangle u = x - 4\langle x, u\rangle u + 4\langle x, u\rangle u\\ &= x,\end{aligned}$$

so $H_u^2 = I$.

Since $H_u^2 = H_u H_u^* = I$, H_u is unitary. □

Exercise 11. Find the spectral decomposition of the following matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Answer. Since the characteristic polynomial of A is

$$f(t) = \det(A - tI) = 4 - 9t + 6t^2 - t^3,$$

$\lambda_1 = 4$ and $\lambda_2 = 1$ are the unique eigenvalues of A. Let g_1 be the linear polynomial such that $g_1(\lambda_j) = \delta_{1j}$, then

$$g_1(x) = \frac{x-1}{4-1} = \frac{x-1}{3} \Rightarrow A_1 = g_1(A) = \frac{A}{3} - \frac{1}{3}I = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$A_2 = I - A_1 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

so the spectral decomposition of A is

$$A = \frac{4}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$