Alex Gittens
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ACM 104: HW 8

Exercise 1. Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator. Suppose that $M_1 \subset H_1$ and $M_2 \subset H_2$ are two subsets such that $T(M_1) \subset M_2$. Prove that $T^*(M_2^\perp) \subset M_1^\perp$.

Proof. Let $x \in M_1$ and $y \in M_2^\perp$, then $Tx \in M_2$, so

$$\langle x, T^*y \rangle_{H_1} = \langle Tx, y \rangle_{H_2} = 0,$$

so $T^*y \in M_1^\perp$. Therefore $T^*(M_2^\perp) \subset M_1^\perp$. □

Exercise 2. Show that $T_0^* = T_0$ and $I^* = I$. (Here: T_0 is the zero operator.)

Proof. Let x, y be elements in the Hilbert space, then

$$\langle T_0 x, y \rangle = 0 = \langle x, T_0 y \rangle$$

and

$$\langle Ix, y \rangle = \langle x, y \rangle = \langle x, Iy \rangle.$$

Since I and T_0 are both bounded linear operators, by the definition of the adjoint, $T_0^* = T_0$ and $I^* = I$. □

Exercise 3. Let T be a bounded linear operator on a Hilbert space H. Show that the range of T is finite-dimensional if and only if T can be written as

$$Tx = \sum_{j=1}^{n} \langle x, v_j \rangle w_j, \quad \forall x \in H,$$

where $v_j, w_j \in H$.

Proof. Assume T can be written in the above form, then $\mathcal{R}(T) = \operatorname{span}(\{w_j\})$, so $\dim(\mathcal{R}(T)) \leq n$; that is, the range of T is finite-dimensional.

Assume the range of T is finite-dimensional, then since $\mathcal{R}(T)$ is an inner product space, it has an orthonormal basis $\{w_j\}_{j=1}^n$. Then

$$Tx = \sum_{j=1}^{n} \langle Tx, w_k \rangle w_k = \sum_{j=1}^{n} \langle x, T^* w_k \rangle w_k,$$

so letting $v_k = T^* w_k$ shows that T can be written in the stated form. □

Exercise 4. Let $\{T_n\}$ be a sequence of normal operators on a Hilbert space H such that $\|T_n - T\| \to 0$ where T is a bounded linear operator on H. Prove that T is normal.

Proof. Let $x, y \in H$, then

$$|\langle (T - T_n x), y \rangle| \leq \|T - T_n\| \|x\| \|y\|,$$

so

$$\langle x, T_n^* y \rangle = \langle T_n x, y \rangle \to \langle Tx, y \rangle = \langle x, T^* y \rangle,$$

showing that $T_n \to T \Rightarrow T_n^* \to T^*$.

Also

$$\|T_n\| \leq \|T_n - T\| + \|T\| \leq 1 + \|T\|$$

for n large enough; consequently $\sup_n \|T_n\|$ is finite. Likewise $\sup_n \|T_n^*\|$ is finite.

Note

$$\|T^*T - T_n^*T_n\| \leq \|T^*T - T_n^*T\| + \|T_n^*T - T_n^*T_n\| \leq \|T\|\|T^* - T_n^*\| + \sup_n (\|T_n^*\|) \|T - T_n\| \to 0$$

and

$$\|TT^* - T_nT_n^*\| \leq \|TT^* - TT_n^*\| + \|TT_n^* - T_nT_n^*\| \leq \|T\|\|T^* - T_n^*\| + \sup_n (\|T_n^*\|) \|T - T_n\| \to 0,$$

so $T_n^*T_n \to T^*T$ and $T_nT_n^* \to TT^*$. Since $T_n^*T_n = T_nT_n^*$, $T^*T = TT^*$, showing T is normal. ☐

Exercise 5. Let T be a bounded linear operator on a complex Hilbert space H. Prove that T is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$.

Proof. Assume T is normal. Let $x \in H$, then

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle x, TT^*x\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle = \|Tx\|^2,$$

so $\|T^*x\| = \|Tx\|$ for all $x \in H$.

Assume $\|T^*x\| = \|Tx\|$ for all $x \in H$, then

$$0 = \langle T^*x, T^*x\rangle - \langle Tx, Tx\rangle = \langle TT^*x, x\rangle - \langle T^*Tx, x\rangle = \langle (TT^* - T^*T)x, x\rangle, \forall x \in H.$$

Since T bounded implies $TT^* - T^*T$ is bounded, and H is complex, this implies $TT^* - T^*T = 0$, so T is normal. ☐

Exercise 6. Let T be a bounded self-adjoint linear operator on a complex Hilbert space H. Show that T is positive if and only if $\sigma(T)$ contains nonnegative real numbers only.

Proof. Note that

$$\langle Tx, x\rangle \geq 0, \forall x \in H \Rightarrow \inf_{\|x\|=1} \langle Tx, x\rangle \geq 0$$

if T is positive. Since

$$\sigma(T) \subset \left[\inf_{\|x\|=1} \langle Tx, x\rangle, \sup_{\|x\|=1} \langle Tx, x\rangle \right],$$

if T is positive, then $\sigma(T)$ contains nonnegative real numbers only.

Likewise, since $\inf_{\|x\|=1} \langle Tx, x\rangle \in \sigma(T)$, if $\sigma(T)$ contains all nonnegative real numbers, then for any $y \in H$

$$\langle Ty, y\rangle = \|y\|^2 \left\langle T\frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle \geq \|y\|^2 \inf_{\|x\|=1} \langle Tx, x\rangle \geq 0,$$

so T is positive.

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