Alex Gittens
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ACM 105: HW 2

Exercise 1. Show that in a Banach space, an absolutely convergent series is convergent.

Proof. Assume $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series in a Banach space, so $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then for each $\epsilon > 0$, there is an integer N such $\sum_{n=N}^{\infty} \|x_n\| < \epsilon$. Therefore, if $m > n > N$,

$$\left\| \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k \right\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=N}^{\infty} \|x_k\| \leq \epsilon,$$

so $\{\sum_{k=1}^{n} x_k\}$ is a Cauchy sequence in the Banach space. By completeness, the series $\sum_{k=1}^{\infty} x_k$ converges. □

Exercise 2.

(a) Let B be the normed linear space of all bounded sequences of complex numbers with norm $\|x\| = \sup |x_j|$ where $x = (x_1, x_2, \ldots)$. Show that the operator $T : B \to B$ defined by $y = (\eta_1, \eta_2, \ldots) = Tx$, $\eta_j = x_j/j$, is linear and bounded.

(b) Show that the range $\mathcal{R}(T)$ is not closed in B.

Proof. (a) Let $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in B$ and $\alpha \in \mathbb{C}$, then

$$T(\alpha x + y) = \left(\frac{\alpha x_1 + y_1}{1}, \frac{\alpha x_2 + y_2}{2}, \ldots \right) = \alpha \left(\frac{x_1}{1}, \frac{x_2}{2}, \ldots \right) + \left(\frac{y_1}{2}, \frac{y_2}{2}, \ldots \right) = \alpha Tx + Ty,$$

so T is linear.

Also, since $|c/j| < |c|$ for all $c \in \mathbb{C}$, $\|Tx\| = \sup_j \left| \frac{x_j}{j} \right| \leq \sup_j |x_j| = \|x\|$, so T is bounded and $\|T\| \leq 1$.

(b) Let $y_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right) \in B$, then $y_n = T(1, 1, \ldots, 1, 0, 0, \ldots)$, so $y_n \in \mathcal{R}(T)$. However, $y_n \to y = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right) \in B$, but there is no $x \in B$ such that $y_n = Tx$, so $y \notin \mathcal{R}(T)$. The existence of a non-convergent Cauchy sequence in $\mathcal{R}(T)$ implies that $\mathcal{R}(T)$ is not complete. Since B is complete and closed subsets of complete spaces are themselves complete, this implies $\mathcal{R}(T)$ is not closed.

□

Exercise 3. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded.

Proof. Let $X = C[0, 1]$ be equipped with the sup norm, and let $T : X \to X$ be the operation of integration,

$$(Tf)(x) = \int_0^x f(s)\, ds.$$

Then T is bounded with norm 1, because

$$|(Tf)(x)| \leq x \sup_{s \in [0,x]} |f(s)| \leq 1 \sup_{s \in [0,1]} |f(s)| \Rightarrow \sup_{x \in [0,1]} |(Tf)(x)| \leq \sup_{s \in [0,1]} |f(s)| \Rightarrow \|Tf\| \leq \|f\|.$$

Then clearly $T^{-1} : \mathcal{R}(T) \to X$ is the linear operation of differentiation,

$$(T^{-1}f)(x) = f'(x).$$

Note $\frac{1}{n} \sin(n\cdot) \to 0$ in Y as $n \to \infty$, but

$$T^{-1} \left(\frac{1}{n} \sin(n\cdot) \right) = \cos(n\cdot) \not\to 0 = T^{-1}(0);$$

that is, T^{-1} is not continuous. Therefore T^{-1} is not bounded.

□

Exercise 4. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt.$$

Proof. Let $x \in C[-1,1]$, then

$$\|f(x)\| = |f(x)| \leq \int_{-1}^{0} |x| \, dt + \int_{0}^{1} |x| \, dt \leq 2 \sup_{t\in[-1,1]} |x(t)| = 2\|x\|,$$

so $\|f\| \leq 2$.

Define the sequence of function x_n given by

$$x_n(t) = \begin{cases} -\operatorname{sgn}(t)\frac{1}{2} & |t| > \frac{1}{n} \\ -t\frac{n}{2} & |t| \leq \frac{1}{n} \end{cases},$$

then for each n, $x_n \in C[-1,1]$ with $\|x_n\| = \frac{1}{2}$ and

$$\|f(x_n)\| = |f(x_n)| = \int_{-1}^{-\frac{1}{n}} \frac{1}{2} \, dt + \int_{-\frac{1}{n}}^{0} -t\frac{n}{2} \, dt - \int_{0}^{\frac{1}{n}} -t\frac{n}{2} \, dt - \int_{\frac{1}{n}}^{1} -\frac{1}{2} \, dt = 2\frac{1}{2}\left(1 - \frac{1}{n}\right) + 2\frac{1}{2}\frac{1}{2}\frac{1}{n} = 1 - \frac{1}{2n},$$

so

$$\|f(x_n)\| = \left(2 - \frac{1}{n}\right)\|x_n\|.$$

Since $2 - \frac{1}{n} \nearrow 2$, this shows that for each $M < 2$, there is an n such that $\|f(x_n)\| > M\|x_n\|$, implying $\|f\| \geq 2$.

Therefore $\|f\| = 2$. □

Exercise 5. If Y is a subspace of a linear space X over K and f is a linear functional on X such that $f(Y) \neq K$, show that $f(y) = 0$ for all $y \in Y$.

Proof. Let X be a linear space, $Y \subset X$ a subspace, and f a linear functional on X. Assume there is a $y \in Y$ such that $f(y) \neq 0$, then for each $k \in K$, $\frac{k}{f(y)}y \in Y$ and

$$f\left(\frac{k}{f(y)}y\right) = \frac{k}{f(y)}f(y) = k,$$

so $f(Y) = K$. Therefore if $f(Y) \neq K$, $f(y) = 0$ for all $y \in Y$. □

Exercise 6. Consider the normed linear space $C[0,1]$ with norm defined by

$$\|x\| = \int_{0}^{1} |x(t)|dt, \quad x \in C[0,1].$$

Let f be a linear functional defined by $f(x) = x(1/2)$. Show that f is not bounded.

Proof. Let $M > 0$, then there is an integer n such that $n > M$. Let x_n be defined by

$$x_n(t) = \begin{cases} 0 & \left|t - \frac{1}{2}\right| \geq \frac{1}{n} \\ n - n^2 t & \left|t - \frac{1}{2}\right| < \frac{1}{n} \end{cases}.$$

Then $x_n \in C[0,1]$ and $\|f(x_n)\| = |x_n(1/2)| = n > M$, and

$$\|x_n\| = \int_{0}^{1} |x_n(t)| \, dt = 2\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} n - n^2 t \, dt = 1.$$

Since for each $M > 0$ there is an x such that $\|x\| = 1$ and $|f(x)| > M$, there is no M such that $\|f(x)\| \leq \|x\|M$ for all $x \in C[0,1]$. Therefore f is not bounded.

□

Exercise 7. Let $f \neq 0$ be a linear functional on a linear space X and let x_0 be a fixed element of $X - \mathcal{N}(f)$. Show that any $x \in X$ has a unique representation $x = \alpha x_0 + y$ where $y \in \mathcal{N}(f)$.

Proof. Let f, X, x_0 be as given. Let $x \in X$, then $\alpha = \frac{f(x)}{f(x_0)}$ is well-defined since $f(x_0) \neq 0$. Then

$$f\left(x - \alpha x_0\right) = f(x) - \frac{f(x)}{f(x_0)} f(x_0) = 0 \Rightarrow y := x - \alpha x_0 \in \mathcal{N}(f),$$

so $x = y + \alpha x_0$ where $y \in \mathcal{N}(f)$.

Assume x is written as $x = y' + \alpha' x_0$ where $y' \in \mathcal{N}(f)$. Then

$$0 = f(x) - f(x) = f(y_1 - y_2) + f((\alpha - \alpha')x_0) = f((\alpha - \alpha')x_0) = (\alpha - \alpha')f(x_0),$$

and since $f(x_0) \neq 0$, $\alpha = \alpha'$. Also, $y = x - \alpha x_0 = x - \alpha' x_0 = y'$, so the representation of x in this form is unique. □