

ACM 105: HW 6

Exercise 1. Let $z \in X$ be a fixed element of a normed linear space X and let $f \in X'$. Show that the operator $T : X \rightarrow X$ defined by $Tx = f(x)z$ is compact.

Proof. Let $\{x_n\}$ be a bounded sequence in X , i.e. $\|x_n\| < M$ for some $M > 0$ and all n . Then $|f(x_n)| \leq \|f\|M$ for all n , so $f(x_n)$ is a bounded sequence in \mathbb{R} . This implies there is a $c \in \mathbb{R}$ and a subsequence x_{n_k} such that $f(x_{n_k}) \rightarrow c$. Therefore $Tx_{n_k} = f(x_{n_k})z \rightarrow cz$, so T maps every bounded sequence in X onto a sequence $\{Tx_n\}$ in X which has a convergent subsequence. That is, T is a compact linear operator. \square

Exercise 2. If a compact linear operator $T : X \rightarrow X$ on an infinite dimensional normed linear space X has an inverse which is defined on all of X , show that the inverse cannot be bounded.

Proof. Assume T^{-1} is defined on all of X and T^{-1} is bounded. Then $x \in B(0; 1) \Rightarrow \|T^{-1}x\| \leq \|T^{-1}\| < \infty$, so $T^{-1}(B(0; 1))$ is a bounded set. Since T is compact, the image $T(T^{-1}(B(0; 1))) = B(0; 1)$ is relatively compact, so $\tilde{B}(0; 1)$ is compact. But this is a contradiction, because X is infinite dimensional, so the closed unit ball is not compact. Therefore the assumption must be incorrect, and T^{-1} is unbounded. \square

Exercise 3. Let $T : X \rightarrow X$ be a compact linear operator on a normed linear space X . If $\dim(X) = \infty$, show that $0 \in \sigma(T)$.

Proof. Assume $0 \in \rho(T)$, then T^{-1} exists, is bounded, and $D(T^{-1})$ is a dense subspace of X (and therefore $\dim D(T^{-1}) = \infty$). Then $TT^{-1} = I_{D(T^{-1})}$, and by Lemma 5.17 (composition of a compact operator and a bounded operator gives a compact operator), this shows $I_{D(T^{-1})}$ is a compact operator on an infinite dimensional subspace of X . But this is a contradiction, because the identity is not a compact operator on infinite dimensional spaces. Therefore $0 \in \sigma(T)$. \square

Exercise 4. Let $E_1 \subset E_2$ be two subsets of \mathbb{R}^n . Show that $|E_1|_e \leq |E_2|_e$.

Answer. Let $S' = \{I_k\}$ be a countable set of intervals covering E_2 , then $E_1 \subset E_2 \subset \cup_{k=1}^{\infty} I_k$, so

$$|E_1|_e = \inf_{S' \text{ covering } E_1} \sigma(S) \leq \sigma(S').$$

This inequality is true for all S' covering E_2 , so

$$|E_1|_e \leq \inf_{S' \text{ covering } E_2} \sigma(S) = |E_2|_e.$$

Exercise 5. Show that the boundary of an interval (that is, closed rectangle) has outer measure zero.

Proof. If $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$, then $I^\circ = (a_1, b_1) \times \cdots \times (a_n, b_n)$. To see this, note that if $x = (x_i)_i$ is in I° , then there is an $r > 0$ such that $B(x; r) \subset I^\circ \subset I$, so in particular, for all i , $a_i \leq x_i - r < x_i < x_i + r \leq b_i \Rightarrow a_i < x_i < b_i$, so the proposed set contains I° . Furthermore, if x is in the proposed set, it satisfies $a_i < x_i < b_i$, so the set $B(x; \min_i \min(|a_i - x_i|, |b_i - x_i|))$ is contained in the proposed set, which is an open set contained in I ; this shows the proposed set is contained in I° . Therefore I° is as proposed.

Also, if $\{x_j = (x_{j,1}, \dots, x_{j,n})\} \subset I$ satisfy $x_j \rightarrow x = (x_1, \dots, x_n)$, then the convergence implies pointwise convergence, and $x_{j,i} \in [a_i, b_i]$ for all j implies $x_{j,i} \rightarrow x_i \in [a_i, b_i] \Rightarrow x \in I$. This shows $\bar{I} \subset I \Rightarrow I = \bar{I}$.

Therefore,

$$\partial I = \bar{I} \setminus I^\circ = [a_1, b_1] \times \cdots \times [a_n, b_n] \setminus (a_1, b_1) \times \cdots \times (a_n, b_n)$$

consists of the points $x = (x_i) \in I$ where $x_i = a_i$ or $x_i = b_i$ for at least one i , or the union

$$\begin{aligned} \partial I = \bigcup_{i=1}^n & \left([a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{i-1}, b_{i-1}] \times a_i \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n] \right. \\ & \left. \cup [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{i-1}, b_{i-1}] \times b_i \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n] \right). \end{aligned}$$

That is, ∂I is the union of 2^n degenerate intervals of the type,

$$I' = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{i-1}, b_{i-1}] \times c \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n]$$

where c is an endpoint of $[a_i, b_i]$.

Note that $|I'|_e = \nu(I') = (b_1 - a_1)(b_2 - a_2) \cdots (c - a_i) \cdots (b_n - a_n) = 0$ by definition for each interval of this type, and one covering of ∂I consists of 2^n intervals of this type, so $|\partial I| \leq 2^n \nu(I') = 0$, showing $|\partial I|_e = 0$. \square

Exercise 6. Show that the empty set has outer measure zero.

Proof. Note that $\emptyset \subset \{0\}$ and $\{0\}$ has outer measure zero because it is a countable set. Therefore by exercise 4 above, $|\emptyset|_e = 0$. \square