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ACM 105: HW 7

Exercise 1. Let $\{E_k\}$ be a sequence of sets with $\sum |E_k|_e < \infty$. Show that $\limsup E_k$ and $\liminf E_k$ have measure zero.

Proof. Since

$$\liminf E_k = \cup_{j=1}^{\infty} \left(\cap_{k=j}^{\infty} E_k \right) \text{ and } \limsup E_k = \cap_{j=1}^{\infty} \left(\cup_{k=j}^{\infty} E_k \right),$$

if $x \in \liminf E_k$, then there is a K such that $x \in \cap_{k=K}^{\infty} E_k$, so that $k > K$ implies $x \in E_k$. Therefore $x \in \cup_{k=j} E_k$ for all j, so $x \in \limsup E_k$; that is, $\liminf E_k \subset \limsup E_k$.

Let $\epsilon > 0$. Note $\sum |E_k|_e < \infty$ implies $\sum_{k>K} |E_k|_e \to 0$ as $K \to \infty$, so there is a K such that $\sum_{k \geq K} |E_k|_e < \epsilon$. Also,

$$x \in \limsup E_k \Rightarrow x \in \cup_{k=K}^{\infty} E_k \Rightarrow \limsup E_k \subset \cup_{k \geq K} E_k,$$

so

$$|\limsup E_k|_e \leq | \cup_{k \geq K} E_k|_e \leq \sum_{k \geq K} |E_k|_e < \epsilon.$$

Since ϵ is arbitrary, $|\limsup E_k|_e = |\limsup E_k| = 0$, and since $\liminf E_k \subset \limsup E_k$, $|\liminf E_k| = 0$ also.

□

Exercise 2. If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. Since $E_2 - E_1$, $E_1 - E_2$, and $E_1 \cap E_2$ are disjoint with union $E_1 \cup E_2$,

$$|E_1 \cup E_2| = |E_1 \cap E_2| + |E_2 - E_1| + |E_1 - E_2|.$$

Also, $E_1 \cap E_2$ and $E_1 - E_2$ are disjoint with union E_1, and $E_1 \cap E_2$ and $E_2 - E_1$ are disjoint with union E_2, so

$$|E_1 \cup E_2| + |E_1 \cap E_2| = (|E_1 \cap E_2| + |E_2 - E_1|) + (|E_1 \cap E_2| + |E_1 - E_2|) = |E_1| + |E_2|.$$

□

Exercise 3. Suppose that $|E|_e < \infty$. Prove that E is measurable if and only if for any $\epsilon > 0$, we have $E = (S \cup N_1) - N_2$, where S is a finite union of non-overlapping intervals, $|N_1|_e < \epsilon$ and $|N_2|_e < \epsilon$.

Proof. Assume E is measurable. Then there is an open $G \supset E$ such that $|G - E| < \epsilon$. Let $N_2 = G - E$.

There is a countable collection of disjoint intervals I_k such that $G = \cup_k I_k$. Let $S_k = \cup_{j=1}^{k} I_j$, then since $S_k \nearrow G$, $\sum_{i=1}^{k} |I_i| = |S_k| \to |G|$, so $\sum_{i>k} |I_i| \to 0$ as $i \to \infty$. Therefore, there is a K such that $|G - S_K| = | \cup_{k>K} I_k| = \sum_{k>K} |I_k| < \epsilon$. Let $S = S_K$ and $N_1 = \cup_{k>K} I_k$.

Then S is a finite union of non-overlapping intervals, $|N_1|_e < \epsilon$, $|N_2|_e < \epsilon$, and

$$E = G - (G - E) = (S \cup N_1) - N_2.$$

Assume for every $\epsilon > 0$ there are S, a finite union of non-overlapping intervals, $|N_1|_e < \epsilon$, and $|N_2|_e < \epsilon$ such that

$$E = (S \cup N_1) - N_2.$$

Given $\epsilon > 0$, choose S, N_1 and N_2 such that $|N_1|_e, |N_2|_e < \epsilon$. Since S is measurable, there is a G_1 such that $G_1 \supset S$ and $|G_1 - S| < \epsilon$. There is also a G_2 such that $G_2 \supset N_1$ and $|G_2| \leq |N_1|_e + \epsilon < 2\epsilon$.

Let $G = G_1 \cup G_2$, then G is open, $G \supset E$, and

$$\begin{aligned} G - E = G_1 \cup G_2 - (S \cup N_1 - N_2) &= ((G_1 \cup G_2) \cap N_2) \cup (G_1 \cup G_2 - (S \cup N_1)) \\ &\subset N_2 \cup (G_1 \cup G_2 - S) \subset N_2 \cup G_2 \cup (G_1 - S) \end{aligned}$$

so

$$|G - E|_e \le |G_1 - S| + |G_2| + |N_2|_e \le \epsilon + 2\epsilon + \epsilon = 4\epsilon.$$

This shows given any $\epsilon > 0$, there is an open set G such that $G \supset E$ and $|G - E|_e < \epsilon$, so E is measurable.

□

Exercise 4. Let $\{E_k\}$ be a sequence of disjoint measurable sets and let A be any set. Show that

$$|A \cap (\cup_{k=1}^{\infty} E_k)|_e = \sum_{k=1}^{\infty} |A \cap E_k|_e.$$

Proof. Clearly

$$|A \cap (\cup_{k=1}^{\infty} E_k)|_e \le \sum_{k=1}^{\infty} |A \cap E_k|_e.$$

Assume A is such that this inequality is strict, then there is a G_δ set H such that $H \supset A \cap (\cup_{k=1}^{\infty} E_k)$ and $|H|_e = |A \cap (\cup_{k=1}^{\infty} E_k)|_e$; since H is G_δ, it is measurable, and the disjoint union of the sets $H \cap E_k$, so

$$|H| = \sum_{k=1}^{\infty} |H \cap E_k|.$$

However, since $A \cap E_k \subset H \cap E_k$,

$$|H| = |A \cap (\cup_{k=1}^{\infty} E_k)|_e < \sum_{k=1}^{\infty} |A \cap E_k|_e \le \sum_{k=1}^{\infty} |H \cap E_k|.$$

This contradiction shows that the desired equality holds for all A. □

Exercise 5. If $E_k \searrow E$ and $|E_1| < \infty$, then $|E| = \lim |E_k|$. Show, by example, that the condition $|E_1| < \infty$ is necessary.

Proof. Let $E_k = [k, \infty)$, then $E_k \supset E_{k+1}$, and $E_k \searrow E = \cap_{k=1}^{\infty} E_k = \emptyset$. Then $|E| = 0$, but for all k,

$$|E_k| = | \cup_{j=k}^{\infty} [j, j+1)| = \sum_{j=k}^{\infty} 1 = \infty.$$

This example shows that there must be some j such that $|E_j| < \infty$ for the theorem to hold. □

Exercise 6. Let E be measurable. Show that $f : E \to \overline{R}$ is measurable if and only if $\{a < f < \infty\}$ is measurable for any finite a.

Proof. Assume f is measurable, then for all integers k, $\{f \ge k\}$ is measurable, so the complement $\{f < k\}$ is measurable, as is the countable union

$$\cup_{k=1}^{\infty} \{f < k\} = \{f < \infty\}.$$

Also, for every finite a, the set $\{f \le a\}$ is measurable, so the complement $\{f > a\}$ is also measurable, as is the intersection

$$\{f < \infty\} \cap \{f > a\} = \{a < f < \infty\}.$$

Assume $\{a < f < \infty\}$ is measurable for any finite a. Assume that, as in the book, the set $\{f = -\infty\}$ is always measurable (otherwise, this theorem is not true since sets of the form $\{a < f < \infty\}$ alone do not generate the Borel σ-field on $\overline{R}$). Then let $a \in \mathbb{R}$. Since by assumption $\{a - \frac{1}{n} < f < \infty\}$ is measurable for all integers n,

$$\{a \leq f < \infty\} = \cap_{n=1}^{\infty} \{a - \frac{1}{n} < f < \infty\}$$

is also measurable, so its complement $\{-\infty \leq f < a\} \cup \{f = \infty\}$ is also measurable. Then letting a vary over the negative integers shows the set

$$\cap_{a=-1}^{-\infty} (\{-\infty \leq f < a\} \cup \{f = \infty\}) = \{f = \pm\infty\}$$

is measurable, so the set $\{f = \infty\} = \{f = \pm\infty\} \cap \{f = -\infty\}^c$ is measurable. Finally, for all finite a, the set

$$\{f > a\} = \{a < f < \infty\} \cup \{f = \infty\}$$

is measurable, so by definition f is measurable.

$\square$