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Theorems from the Book

Theorem 1 (3.25)**.** *(a) If* $|a_n| \le c_n$ *for* $n \ge N_0$*, where* N_0 *is some fixed integer, and if* $\sum c_n$ *converges, then* $\sum a_n$ *converges.*
(b) If $a_n \ge d_n \ge 0$ *for* $n \ge N_0$*, and if* $\sum d_n$ *diverges, then* $\sum a_n$ *diverges.*

Theorem 2 (3.28)**.** $\sum \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \le 1$*.*

Theorem 3 (7.10)**.** *Suppose* $\{f_n\}$ *is a sequence of functions defined on* E*, and suppose*

$$|f_n(x)| \le M_n \quad (x \in E, n = 1, 2, \ldots).$$

Then $\sum f_n$ *converges uniformly on* E *if* $\sum M_n$ *converges.*

Problems

Page 165, Problem 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let $\{f_n\}$ be a sequence of functions on E which uniformly converge to f, with f_n bounded by M_n. Let $\epsilon = \frac{1}{2}$, then there is an $N \in \mathbb{Z}^+$ such that $n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon$ for all $x \in E$. Therefore, $|f(x)| < \epsilon + |f_N(x)| < \epsilon + M_N = M_b$ for all $x \in E$. Furthermore, $|f(x) - f_n(x)| < \epsilon \Rightarrow |f_n(x)| < \epsilon + |f(x)| < \epsilon + M_b = 1 + M_N$, for all $x \in E$ and all $n \ge N$. Since $|f_n(x) - f(x)| < M_n + M_b$, $\sup_{x\in E} |f_n(x) - f(x)| \in \mathbb{R}$. Let $d = \max\{\sup_{x\in E} |f_1(x) - f(x)|, \ldots, \sup_{x\in E} |f_{N-1}(x) - f(x)|\}$. For $n < N$ and all $x \in E$,

$$|f_n(x)| \le |f_n(x) - f(x)| + |f(x)| < \sup_{x\in E} |f_n(x) - f(x)| + M_b \le d + M_b.$$

Then for all n, $|f_n(x)| < \max\{1 + M_N, d + M_b\}$. Therefore $\{f_n\}$ is uniformly bounded.

□

Page 165, Problem 2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof. Let $\{f_n\}$ converge uniformly to f and $\{g_n\}$ converge uniformly to g, with f_n and g_n defined on the set E. Then for all $\epsilon > 0$, there are $N_f, N_g \in \mathbb{Z}^+$ such that

$$n \ge N_f \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ and } n \ge N_g \Rightarrow |g_n(x) - g(x)| < \frac{\epsilon}{2},$$

for all $x \in E$. Let $N_s = \max\{N_f, N_g\}$, then for all $n \ge N_s$ and all $x \in E$,

$$|(f_n(x) + g_n(x)) - (f(x) + g(x))| \le |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon,$$

therefore $\{f_n + g_n\}$ converges uniformly to $\{f + g\}$.

Furthermore, if f_n is bounded by F_n and g_n is bounded by G_n, there are $N_f, N_g \in \mathbb{Z}^+$ such that

$$n \geq N_f \Rightarrow |f_n(x) - f(x)| < \frac{1}{2} \Rightarrow |f(x)| < |f_{N_f}(x)| + \frac{1}{2} = M_f$$
$$n \geq N_g \Rightarrow |g_n(x) - g(x)| < \frac{1}{2} \Rightarrow |g(x)| < |g_{N_g}(x)| + \frac{1}{2} = M_g,$$

for all $x \in E$. Also,

$$|f_n(x)g_n(x) - f(x)g(x)| = |(g_n(x) - g(x))(f_n(x) - f(x)) + g(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))|,$$

so

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |(g_n(x) - g(x))(f_n(x) - f(x))| + |g(x)(f_n(x) - f(x))| + |f(x)(g_n(x) - g(x))|.$$

Given $\epsilon > 0$, choose $\epsilon_0 > 0$ such that $\epsilon_0^2 + (M_g + M_f)\epsilon_0 < \epsilon$. By the uniform convergence of $\{f_n\}$ and $\{g_n\}$, there is an N such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon_0 \text{ and } |g_n(x) - g(x)| < \epsilon_0$$

for all $x \in E$. Therefore for $n > N$,

$$|f_n(x)g_n(x) - f(x)g(x)| \leq \epsilon_0^2 + (M_g + M_f)\epsilon_0 < \epsilon,$$

so $\{f_n g_n\}$ converges uniformly to $\{fg\}$. ☐

Page 166, Problem 6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. Let

$$f_n = (-1)^{2n-1}\frac{x^2 + 2n - 1}{(2n-1)^2} + (-1)^{2n}\frac{x^2 + 2n}{(2n)^2}$$

and

$$S(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \frac{-\left((4n-1)x^2 + 2n(2n-1)\right)}{4n^2(2n-1)^2}$$

be defined on an interval E contained in the interval $[a, b]$, and let $c = \max\{|a|, |b|\}$. Then there is a $N \in \mathbb{Z}^+$, $N > 1$, such that $n \geq N \Rightarrow x^2 < c^2 < n$, so for all $x \in E$ and $n \geq N$,

$$|f_n(x)| = \frac{(4n-1)x^2 + 2n(2n-1)}{4n^2(2n-1)^2} < \frac{(4n-1)n + 2n(2n-1)}{4n^2(2n-1)^2}$$
$$= \frac{8n^2 - 3n}{16n^4 - 16n^3 + 4n^2} < \frac{8n^2}{16n^4 - 16n^3} = \frac{1}{2n(n-1)} = M_n.$$

For $n < N$, let $M_n = (4n-1)c^2 + 2n(2n-1)$, so $|f_n(x)| < M_n$ for all $x \in E$ and $n \in \mathbb{Z}^+$.

There is a $N_0 \in \mathbb{Z}^+$ such that for $n \geq N_0$, $n^2 - 3n + 1 > 0$, so for $n \geq N_0$,

$$n^2 - 2n + 1 > n \Rightarrow (n-1)^2 > n \Rightarrow \sqrt{n} < n - 1 \Rightarrow \frac{1}{2n(n-1)} < \frac{1}{2n^{1.5}} \Rightarrow M_n < \frac{1}{2n^{1.5}}.$$

Let $c_n = \frac{1}{2n^{1.5}}$, so by Theorem 2, $\sum c_n$ converges, and by Theorem 1, $\sum M_n$ converges. Therefore by Theorem 3, $S(x) = \sum f_n$ uniformly converges on the interval E.

Let

$$g_n = \left|(-1)^n \frac{x^2+n}{n^2}\right| = \frac{x^2}{n^2} + \frac{1}{n},$$

and $c_n = \frac{1}{n}$, so that $g_n \geq c_n$ for all $x \in \mathbb{R}$. By Theorem 2, $\sum c_n$ diverges, and by Theorem 1, $\sum g_n$ diverges also. Therefore, the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

does not converge absolutely.

□