ANALYSIS HW 9

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Problems

Exercise 9, pg. 290. Define $(x, y) = T(r, \theta)$ on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Show that T maps this rectangle onto the closed disc D with center at $(0, 0)$ and radius a, that T is one-to-one in the interior of the rectangle, and that $J_T(r, \theta) = r$. If $f \in \mathcal{C}(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x, y)\, dx\, dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r\, dr\, d\theta.$$

Hint: Let D_0 be the interior of D, minus the interval from $(0, 0)$ to $(0, a)$. As it stands, Theorem 10.9 applies to continuous functions f whose support lies in D_0. To remove this restriction, proceed as in Example 10.4.

Proof. Let $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$, then $T(r, \theta) = (x, y) = (r\cos\theta, r\sin\theta)$ satisfies $x^2 + y^2 = r^2 \leq a^2$, so T maps into D. Assume $(x, y) \neq (0, 0) \in D$, then $r = \sqrt{x^2 + y^2}$ and

$$\theta = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & x < 0 \end{cases}$$

satisfy $x = r\cos\theta$ and $y = r\sin\theta$; also, $(x, y) = (0, 0) = T(0, 0)$, so T is surjective on D. Therefore T maps onto D.

Assume $T(r_1, \theta_1) = (x, y) = T(r_2, \theta_2)$ where $(r_1, \theta_1), (r_2, \theta_2)$ are in the interior of the rectangle, so $0 < \theta_1, \theta_2 < 2\pi$. Then

$$r_2 = \sqrt{r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2} = \sqrt{x^2 + y^2} = \sqrt{r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta_1} = r_1,$$

so

$$x = r_1\cos\theta_1 = r_2\cos\theta_2 \Rightarrow \cos\theta_1 = \cos\theta_2$$
$$y = r_1\sin\theta_1 = r_2\sin\theta_2 \Rightarrow \sin\theta_1 = \sin\theta_2;$$

since $0 < \theta_1, \theta_2 < 2\pi$, these equalities imply $\theta_1 = \theta_2$. Therefore T is 1-1 on the interior of the rectangle.

Since

$$T'(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

$J_T(r, \theta) = r\cos^2\theta + r\sin^2\theta = r.$

Let $D_0 = T((0,a)\times(0,2\pi))$, i.e., the interior of D minus the interval from $(0,0)$ to $(0,a)$. Notice that $J_T \neq 0$ in D_0, since $r \neq 0$ in D_0.

Let $R_n = \left(0, a\left(1-\frac{1}{n}\right)\right)\times(0,2\pi)$ and $D_n = T(R_n)$ for $n = 2,3,4,\ldots$; notice $\overline{D_n} \subset D_0$. Construct a function ψ_n such that ψ_n is continuous on $\mathbb{R}^2$, $\psi_n = 1$ on $\overline{D_n}$, $\psi_n = 0$ on $\mathbb{R}^2 - D_n$, and $0 \leq \psi_n \leq 1$ on $\mathbb{R}^2$. Let $f_n = \psi_n f$ so that $f_n \in \mathcal{C}(D)$. Since f_n has support in $D_n \subset D_0$, by the Change of Variables Theorem,

$$\int_D f_n(x,y)\,dx\,dy = \int_0^a \int_0^{2\pi} f_n(T(r,\theta))r\,dr\,d\theta.$$

Since $f \in \mathcal{C}(D)$, there is an $M > 0$ such that $|f| < M$ on D; therefore, for $n = 2,3,4,\ldots$,

$$\begin{aligned}\left|\int_D f(x,y)\,dx\,dy - \int_D f_n(x,y)\,dx\,dy\right| &= \left|\int_D (1-\psi_n)f(x,y)\,dx\,dy\right| \\ &\leq M(A(D) - A(D_n)) = M\left(\pi a^2 - \pi\left(a\left(1-\frac{1}{n}\right)\right)^2\right) \\ &= \frac{Ma^2\pi(2n-1)}{n^2},\end{aligned}$$

where $A(D)$ and $A(D_n)$ denote the area of the regions D and D_n, respectively.

Also,

$$\begin{aligned}\left|\int_0^a\int_0^{2\pi} f(T(r,\theta))r\,dr\,d\theta - \int_0^a\int_0^{2\pi} f_n(T(r,\theta))r dr\,d\theta\right| &\leq M\left(2\pi a - 2\pi\left(a\left(1-\frac{1}{n}\right)\right)\right) \\ &= \frac{M2\pi a}{n},\end{aligned}$$

so by the triangle inequality,

$$\left|\int_D f(x,y)\,dx\,dy - \int_0^a\int_0^{2\pi} f(T(r,\theta))r\,dr\,d\theta\right| \leq \frac{M2\pi a}{n} + \frac{Ma^2\pi(2n-1)}{n^2},$$

for $n = 2,3,4,\ldots$. Clearly, for all $\epsilon > 0$, there exist an N such that $n > N$ implies

$$\left|\int_D f(x,y)\,dx\,dy - \int_0^a\int_0^{2\pi} f(T(r,\theta))r\,dr\,d\theta\right| < \epsilon,$$

therefore

$$\int_D f(x,y)\,dx\,dy = \int_0^a\int_0^{2\pi} f(T(r,\theta))r\,dr\,d\theta.$$

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Exercise 15, pg. 291. If ω and λ are k- and m-forms, respectively, prove that

$$\omega\wedge\lambda = (-1)^{km}\lambda\wedge\omega.$$

Proof. Let $\omega = \sum_I b_I(\mathbf{x})\,dx_I$ and $\lambda = \sum_J c_J(\mathbf{x})\,dx_J$ be k- and m-forms, respectively, in their standard presentation, where $I = \{i_1,\ldots,i_k\}, J = \{j_1,\ldots,j_k\}$ are increasing k- and m-indices, respectively. Either $I\cap J \neq \emptyset$, in which case $dx_I\wedge dx_J = 0 = (-1)^{km}dx_J\wedge dx_I$ trivially, or $I\cap J = \emptyset$.

In the latter case, let $1 \leq p \leq m$ and consider the statement

$$dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_p}\wedge dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_{p+1}}\wedge\cdots\wedge dx_{j_m} = (-1)^{kp}dx_I\wedge dx_J.$$

This is true for $p = 1$, because it takes k interchanges to move the term dx_{j_1} to the left of all the terms in the basic k-form dx_I, and each interchange introduces a factor of -1. Assume it is true for p, so

$$dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_p} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_{p+1}} \wedge \cdots \wedge dx_{j_m} = (-1)^{kp} dx_I \wedge dx_J;$$

it takes k more interchanges to move the term $dx_{j_{p+1}}$ to the immediate right of the term dx_{j_p} (i.e., to the left of all the terms in the basic k-form dx_I), so

$$dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_p} \wedge dx_{j_{p+1}} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_{p+2}} \wedge \cdots \wedge dx_{j_m} = (-1)^{k(p+1)} dx_I \wedge dx_J,$$

so it is also true for $p + 1$. By induction, this statement is true for $p = m$, so

$$dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I.$$

Therefore,

$$\begin{aligned} \omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) \; dx_I \wedge dx_J \\ &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) (-1)^{km} dx_J \wedge dx_I \\ &= (-1)^{km} \lambda \wedge \omega \end{aligned}$$

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