

ANALYSIS HW 4

ALEX GITTENS

PROBLEMS

Problem 7, page 197. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Proof. Let $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x$. Since $0 < \cos x < 1$ for $x \in (0, \frac{\pi}{2})$, $f'(x) > 0$, so $f(x)$ is increasing on $(0, \frac{\pi}{2})$. Also, $f(0) = 0$, so $x > \sin x$ for $x \in (0, \frac{\pi}{2})$. Therefore

$$\frac{\sin x}{x} < 1$$

for $x \in (0, \frac{\pi}{2})$.

Let $h(x) = \frac{\sin x}{x}$, then

$$h'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2}.$$

Let $g(x) = x - \tan x$, then $g'(x) = 1 - \sec^2 x$, and $0 < \cos x < 1$ for $x \in (0, \frac{\pi}{2})$, so $\sec^2 x > 1$, and $g'(x) < 0$ for $x \in (0, \frac{\pi}{2})$. Also, $g(0) = 0$, so $g(x)$ is decreasing on $x \in (0, \frac{\pi}{2})$, and $x < \tan x$.

Therefore, $h'(x) < 0$, so $h(x)$ is decreasing on $(0, \frac{\pi}{2})$. Thus, for $x \in (0, \frac{\pi}{2})$,

$$\frac{\sin x}{x} > f\left(\frac{\pi}{2}\right) = \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

□

Problem 8, page 197. For $n = 0, 1, 2, \dots$, and x real, prove that

$$|\sin nx| \leq n|\sin x|.$$

Note that this inequality may be false for other values of n . For instance,

$$\left|\sin \frac{1}{2}x\right| \geq \frac{1}{2}|\sin \pi|.$$

Proof. Notice

$$|\sin nx| = \left| \frac{e^{inx} - e^{-inx}}{2i} \right| = \frac{1}{2} |e^{inx}(1 - e^{-i2nx})| = \frac{1}{2} |1 - e^{-i2nx}|$$

and

$$|\sin x| = \left| \frac{e^{ix} - e^{-ix}}{2} \right| = \frac{1}{2} |e^{ix}(1 - e^{-i2x})| = \frac{1}{2} |1 - e^{-i2x}|.$$

Therefore,

$$\frac{|\sin nx|}{|\sin x|} = \left| \frac{1 - e^{-i2nx}}{1 - e^{-i2x}} \right| = \left| \sum_{k=0}^{n-1} e^{-i2kx} \right| \leq \sum_{k=0}^{n-1} |e^{-i2kx}| = n,$$

so

$$|\sin nx| \leq n|\sin x|.$$

□