

## ANALYSIS HW 5

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### PROBLEMS

**Exercise 13, p. 198.** Put  $f(x) = x$  if  $0 \leq x < 2\pi$ , and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

*Proof.* Let  $f(x) = x$  for  $x \in [0, 2\pi)$  and  $f(x + 2\pi) = f(x)$  for  $x \in \mathbb{R}$ . Then  $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$ , where

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx.$$

For  $m \neq 0$ ,

$$2\pi c_m = \int_{-\pi}^{\pi} f(x) e^{-imx} dx = \int_{-\pi}^0 (2\pi + x) e^{-imx} dx + \int_0^{\pi} x e^{-imx} dx = \frac{2\pi}{-im}$$

and

$$2\pi c_0 = \int_{-\pi}^{\pi} f(x) dx = \int_0^{2\pi} x dx = 2\pi^2,$$

so

$$\sum_{m=-\infty}^{\infty} |c_m|^2 = 2 \sum_{m=1}^{\infty} |c_m|^2 + |c_0|^2 = 2 \sum_{m=1}^{\infty} \frac{1}{m^2} + \pi^2.$$

Clearly  $f$  is  $2\pi$ -periodic and Riemann integrable, so by Parseval's theorem,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \frac{4\pi^2}{3} \\ &= 2 \sum_{m=1}^{\infty} \frac{1}{m^2} + \pi^2, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\frac{4\pi^2}{3} - \pi^2}{2} = \frac{\pi^2}{6}.$$

□

**Exercise 16, p. 199.** Prove a pointwise version of Fejér's theorem: If  $f \in \mathcal{R}$  and  $f(x+), f(x-)$  exist for some  $x$ , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2} [f(x+) + f(x-)].$$

*Proof.* Given  $x$  and  $\epsilon > 0$ , let  $h(x) = \frac{1}{2} (f(x+) + f(x-))$ ; consider

$$\begin{aligned} |\sigma_n(f; x) - h(x)| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - h(x)) K_n(t) dt \right| \\ &= \frac{1}{2\pi} \left| \int_{-\pi}^0 (f(x-t) - h(x)) K_n(t) dt + \int_0^{\pi} (f(x-t) - h(x)) K_n(t) dt \right| \\ &= \frac{1}{2\pi} \left| - \int_{\pi}^0 (f(x+t) - h(x)) K_n(t) dt + \int_0^{\pi} (f(x-t) - h(x)) K_n(t) dt \right| \\ &= \frac{1}{2\pi} \left| \int_0^{\pi} [f(x+t) + f(x-t) - 2h(x)] K_n(t) dt \right| \end{aligned}$$

where the first equality holds because  $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t) dt = 1$ , and the others hold because  $K_n(t)$  is an even function.

Since  $f(x-t) \rightarrow f(x-)$  and  $f(x+t) \rightarrow f(x+)$  as  $t \rightarrow 0+$ , there is a  $\delta > 0$  such that  $t \in (0, \delta) \Rightarrow |f(x+t) + f(x-t) - f(x+) - f(x-)| < \frac{\epsilon}{2}$ . Therefore

$$\begin{aligned} |\sigma_n(f; x) - h(x)| &\leq \frac{1}{2\pi} \int_0^{\delta} |f(x+t) + f(x-t) - 2h(x)| K_n(t) dt \\ &\quad + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x+t) + f(x-t) - 2h(x)| K_n(t) dt \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\epsilon}{2} K_n(t) dt + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x+t) + f(x-t) - 2h(x)| K_n(t) dt \\ &= \frac{\epsilon}{2} + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x+t) + f(x-t) - 2h(x)| K_n(t) dt, \end{aligned}$$

where the fact that  $K_N(t) \geq 0$  allows it to be kept outside the absolute value sign and allows the extension of the limits in the first integral in the second inequality.

Since  $f \in \mathcal{R}$ ,  $f$  is bounded, so there exists an  $M > 0$  such that  $|f(x+t) + f(x-t) - 2h(x)| < M$  for all  $t \in (0, \pi]$ . Furthermore, it is a property of  $K_n(t)$  that if  $0 < \delta \leq t \leq \pi$ ,

$$K_n(t) \leq \frac{1}{n+1} \cdot \frac{2}{1 - \cos \delta},$$

so

$$|\sigma_n(f; x) - h(x)| \leq \frac{\epsilon}{2} + \frac{1}{\pi(n+1)} \int_{\delta}^{\pi} \frac{M}{1 - \cos \delta} dt = \frac{\epsilon}{2} + \frac{M(\pi - \delta)}{\pi(n+1)(1 - \cos \delta)}.$$

Clearly, there is an  $N$  such that  $n \geq N$  implies the last term is less than  $\frac{\epsilon}{2}$ . Therefore  $|\sigma_n(f; x) - h(x)| \rightarrow 0$  as  $n \rightarrow \infty$ , or

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2} [f(x+) + f(x-)].$$

□