ANALYSIS HW 8

ALEX GITTENS

Problems

Exercise 16, pg. 240. Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded on $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

Proof. By definition,

$$f'(0) = \lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{t + 2t^2 \sin\left(\frac{1}{t}\right)}{t} = 1 + 2\lim_{t\to 0} t \sin\left(\frac{1}{t}\right).$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \to 0$ and $t_n \neq 0$ for all n, then given $\epsilon > 0$, there is an N such that $|t_n| < \epsilon$ for $n > N$, therefore

$$\left|0 - t_n \sin\left(\frac{1}{t_n}\right)\right| = |t_n| \left|\sin\left(\frac{1}{t_n}\right)\right| \leq |t_n| < \epsilon$$

for $n > N$. Therefore

$$\lim_{t\to 0} t \sin\left(\frac{1}{t}\right) = 0,$$

and $f'(0) = 1$.

When $t \neq 0$, $|f'(t)| = \left|1 + 4t \sin\left(\frac{1}{t}\right) - 2\cos\left(\frac{1}{t}\right)\right| \leq 1 + 4|t| \left|\sin\left(\frac{1}{t}\right)\right| + 2\left|\cos\left(\frac{1}{t}\right)\right|$, and $\left|\sin\left(\frac{1}{t}\right)\right| \leq 1$, $\left|\cos\left(\frac{1}{t}\right)\right| \leq 1$, so when $t \in (-1,1)\backslash\{0\}$,

$$|f'(t)| \leq 1 + 4 + 2 = 7,$$

therefore $f'(t)$ is bounded on $(-1, 1)$.

Given $\delta > 0$, take $n \in \mathbb{N}$ such that $t_1 = \frac{1}{2\pi n + \frac{\pi}{2}}$ and $t_2 = \frac{1}{2\pi n}$ satisfy $0 < t_1 < t_2 < \delta$, then

$$f'(t_1) = 1 + 4t_1 > 0$$
$$f'(t_2) = 1 - 2 = -1 < 0,$$

therefore, since f' is continuous, there is a $t' \in (t_1, t_2)$ such that $f'(t) = 0$. That is, f has a local maximum or minimum at t'. Without loss of generality, take $f(t')$ to be a local maximum. Then there exist t_l, t_r satisfying the relations $t_1 < t_l < t' < t_r < t_2$ such that $f(t') > f(t_l)$ and $f(t') > f(t_r)$. Let $v = \max f(t_l), f(t_r)$, then by the intermediate value theorem, there is an a satisfying $t_l \leq a < t'$ and a b satisfying $t' < b \leq t_r$ such that $f(a) = f(b) = v$.

That is, in any δ neighborhood of 0, there exist two numbers a and b satisfying $a \neq b$ and $f(a) = f(b)$, so f is not one-to-one in any δ neighborhood of 0. □

Exercise 19, pg. 240. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Proof. Let $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^3$ be defined as

$$\mathbf{f}(x_1, x_2, x_3, x_4) = (3x_1 + x_2 - x_3 + x_4^2, x_1 - x_2 + 2x_3 + x_4, 2x_1 + 2x_2 - 3x_3 + 2x_4),$$

and $(x, y, z, u) \in \mathbb{R}^4$ be such that

$$\mathbf{f}(x, y, z, u) = \mathbf{0}.$$

Then

$$\mathbf{f}'(x_1, x_2, x_3, x_4) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}$$

Taking

$$A_x = \begin{pmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix},$$

and noticing $\det(A_x) = 8u - 12$ so that A_x is invertible when $u \neq \frac{3}{2}$, it is clear from the Implicit Function Theorem that there is a function $g : \mathbb{R} \to \mathbb{R}^3$ that determines (x, y, u) as a function of z in a neighborhood of the point (x, y, z, u) when $u \neq \frac{3}{2}$. If $u = \frac{3}{2}$, then the only solution of the system of equations is $(x = \frac{-21}{40}, y = \frac{-3}{40}, z = \frac{3}{5})$, therefore if (x, y, z, u) is a solution of the system of equations, (x, y, u) are always solvable (locally) in terms of z.

Taking

$$A_x = \begin{pmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix},$$

and noticing $\det A_x = 21 - 14u$ so that A_x is invertible when $u \neq \frac{3}{2}$, it is clear from the Implicit Function Theorem that there is a function $g : \mathbb{R} \to \mathbb{R}^3$ that determines (x, z, u) as a function of y in a neighborhood of the point (x, y, z, u) when $u \neq \frac{3}{2}$. If $u = \frac{3}{2}$, then the only solution of the system of equations is $(x = \frac{-21}{40}, y = \frac{-3}{40}, z = \frac{3}{5})$, therefore if (x, y, z, u) is a solution of the system of equations, (x, z, u) are always solvable (locally) in terms of y.

Taking

$$A_x = \begin{pmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix},$$

and noticing $\det A_x = 3 - 2u$ so that A_x is invertible when $u \neq \frac{3}{2}$, it is clear from the Implicit Function Theorem that there is a function $g : \mathbb{R} \to \mathbb{R}^3$ that determines (y, z, u) as a function of x in a neighborhood of the point (x, y, z, u) when $u \neq \frac{3}{2}$. If $u = \frac{3}{2}$, then the only solution of the system of equations is $(x = \frac{-21}{40}, y = \frac{-3}{40}, z = \frac{3}{5})$, therefore if (x, y, z, u) is a solution of the system of equations, (y, z, u) are always solvable (locally) in terms of x.

Taking

$$A_x = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3, \end{pmatrix}$$

and noticing $\det A_x = 0$ so that A_x is not invertible, it is clear the Implicit Function Theorem does not apply. It is possible that there is no function $g : \mathbb{R} \to \mathbb{R}^3$ that determines (x, y, z) as a function of u in the neighborhood of a solution to the system of equations. □

Exercise 23, pg. 241. Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$, $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Proof. Clearly

$$f(0, 1, -1) = 0^2 \times 1 + e^0 + -1 = 1 - 1 = 0,$$

and

$$(D_1 f)(x_1, x_2, x_3) = 2x_1 + e^{x_1},$$

so

$$(D_1 f)(0, 1, -1) = 2 \times 0 + e^0 = 1 \neq 0.$$

Clearly the partial derivatives of f are continuous, therefore $f \in C'(-1, 1)$. Let $E = (-1, 1)$ and $A_x = f'(0, -1, -1)$, then A_x is invertible with $A_x^{-1} = 1$, so by the Implicit Function Therorem, there are open sets $U \subset \mathbb{R}^3$ and $W \subset R^2$ with $(0, 1, -1) \in U$ and $(1, -1) \in W$ and a C' map $g : W \to \mathbb{R}^3$ such that $g(1, -1) = 0$ and $f(g(\mathbf{y}), \mathbf{y}) = f(g(y_1, y_2), y_1, y_2) = 0$ for $\mathbf{y} = (y_1, y_2) \in W$.

Letting $A_y = [D_2 f, D_3 f] = [x_1^2, 1]$, by the Implicit Function Theorem,

$$g'(x_2, x_3] = -A_x^{-1} A_y = [-D_2 f(x_1, x_2, x_3), -D_3 f(x_1, x_2, x_3] = [-x_1^2, -1],$$

so

$$(D_1 g)(1, -1) = -0^2 = 0 \text{ and } (D_2 g) = -1.$$

□