ANALYSIS HW 9

ALEX GITTENS

Problems

Exercise 5, pg. 289. Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the function φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap. 4.)

Lemma 1. *If* (X, d) *is a metric space,* $A \subset X$*, and* $d_A : X \to \mathbb{R}^+$ *is defined by* $d_A(x) = \inf_{a \in A} d(x, a)$*, then* d_A *is continuous on* X*.*

Furthermore, if $A, B \subset X$ *are closed and nonempty in* X*, and* $A \cap B = \emptyset$*, and*

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}, \quad \forall x \in X,$$

then $f(A) = \{0\}$*,* $f(B) = \{1\}$*,* f *is continuous on* X*, and* $0 \leq f(x) \leq 1$ *for all* $x \in X$*.*

Proof. Notice that for all $x \in X$, $a \in A$, $d(x, a)$ is bounded below by 0, so d_A is well defined on X. Let $x \in X$; then there exists a sequence $\{a_n\} \subseteq A$ such that $d(x, a_n) \to d_A(x)$ as $n \to \infty$. Let $x' \in X$, then

$$d_A(x') \leq d(x', a_n) \leq d(x', x) + d(x, a_n) \quad \forall n \in \mathbb{N},$$

so

$$d_A(x') \leq d(x', x) + d_A(x).$$

Likewise,

$$d_A(x) \leq d(x', x) + d_A(x'),$$

so

$$d_A(x) - d(x', x) \leq d_A(x') \leq d_A(x) + d(x', x).$$

Let $\{x'_n\} \subseteq X$ be a sequence such that $x'_n \to x$, then by the above,

$$d_A(x'_n) \to d_A(x),$$

therefore d_A is continuous in X.

If A is a closed subset of X and $x \in X$ is such that $d_A(x) = 0$, then there exists a sequence $\{a_n\} \subseteq A$ such that $a_n \to x$, so $x \in A$. If A and B are closed disjoint sets, then $x \in A \to x \notin B$ and vice versa, so $d_A(x) + d_B(x) \neq 0$ for all $x \in X$. Therefore

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}, \quad \forall x \in X$$

is continuous on X.

Furthermore, if $x \in A$, then $d_A(x) \leq d(x,x) = 0$, so $f(x) = 0$. If $x \in B$, then $d_B(x) = 0$ and $d_A(x) \neq 0$, so $f(x) = 1$. Clearly, for all $x \in X$, $d_A(x) \leq d_A(x) + d_B(x)$ and both the numerator and denominator are positive, so $0 \leq f(x) \leq 1$. ☐

Theorem 1. *Suppose K is a compact subset of a metric space X, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \ldots, \psi_s \in C(X)$ such that*

(a) $0 \leq \psi_i \leq 1$ for $1 \leq i \leq s$;
(b) each ψ_i has its support in some V_α, and
(c) $\psi_1(x) + \cdots + \psi_s(x) = 1$ for every $x \in K$.

Proof. Associate with each $\mathbf{x} \in K$ an index $\alpha(\mathbf{x})$ so that $\mathbf{x} \in V_{\alpha(\mathbf{x})}$. Then there are open balls $B(\mathbf{x})$ and $W(\mathbf{x})$, centered at $\mathbf{x}$, with

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset V_{\alpha(\mathbf{x})}.$$

Since K is compact, there are points $\mathbf{x}_1, \ldots, \mathbf{x}_s$ in K such that

$$K \subset B(\mathbf{x}_1) \cup \cdots \cup B(\mathbf{x}_s).$$

By Lemma 1, there are functions $\varphi_1, \ldots, \varphi_s \in C(X)$, such that $\varphi_i(\mathbf{x}) = 1$ on $\overline{B(\mathbf{x}_i)}$, $\varphi_i(\mathbf{x}) = 0$ outside $W(\mathbf{x}_i)$ (since $K - W(\mathbf{x}_i)$ is a closed set disjoint with $\overline{B(\mathbf{x}_i)}$), and $0 \leq \varphi_i(\mathbf{x}) \leq 1$ on X.

Notice that for all $\mathbf{x} \in K$, $\sum \varphi_j(\mathbf{x}) \neq 0$, because by the non-negativity of the φ_js, equality would imply $\forall j : \varphi_j(\mathbf{x}) = 0$, which would imply $\forall j : \mathbf{x} \notin B(\mathbf{x}_j)$. This is a contradiction, since $\{B(\mathbf{x}_j)\}$ covers K.

Let

$$\psi_i = \frac{\varphi_i}{\sum_{j=1}^s \varphi_j}, \quad 1 \leq i \leq s,$$

then for all $\mathbf{x} \in K$,

$$\sum_{i=1}^s \psi_i(\mathbf{x}) = \frac{\sum_{i=1}^s \varphi_i(\mathbf{x})}{\sum_{j=1}^s \varphi_j(\mathbf{x})} = 1,$$

so (c) is satisfied. Clearly (b) is satisfied, and (a) is satisfied, because

$$0 \leq \psi_i = \frac{\varphi_i}{\varphi_i + \sum_{j \neq i} \varphi_j} \leq \varphi_i \leq 1, \quad 1 \leq i \leq s.$$

☐

Exercise 6, pg. 289. Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 1 of Chap. 8 in the construction of the auxiliary functions φ_i.)

Lemma 2. *The function*

$$b(\mathbf{x}) = \begin{cases} e^{\frac{-1}{1-\|\mathbf{x}\|^2}} & |\mathbf{x}| < 1 \\ 0 & |\mathbf{x}| \geq 1 \end{cases}$$

defined on $\mathbb{R}^n$ satisfies:

(a) $b \in C^\infty(\mathbb{R}^n)$
(b) $0 < b(\mathbf{x}) \leq 1$ when $|\mathbf{x}| < 1$

(c) $b(\mathbf{x}) = 0$ *when* $|\mathbf{x}| \geq 1$

Proof. Clearly, by definition b satisfies (b) and (c).

Let

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Then $b(\mathbf{x}) = f(1 - \|\mathbf{x}\|^2)$ and clearly $1 - \|\mathbf{x}\|^2$ is $C^\infty(\mathbb{R}^n)$, so by the Chain Rule, $b \in C^\infty(\mathbb{R}^n)$ if $f \in C^\infty(\mathbb{R})$.

By Exercise 1, Chapter 8 (attached), $f \in C^\infty(\mathbb{R})$, so $b \in C^\infty(\mathbb{R}^n)$.

□

Theorem 2. *Suppose K is a compact subset of $\mathbb{R}^n$, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \ldots, \psi_s \in C^\infty(\mathbb{R}^n)$ such that*

(a) $0 \leq \psi_i \leq 1$ *for* $1 \leq i \leq s$*;*
(b) each ψ_i *has its support in some* V_α*, and*
(c) $\psi_1(x) + \cdots + \psi_s(x) = 1$ *for every* $x \in K$*.*

Proof. Associate with each $\mathbf{x} \in K$ an index $\alpha(\mathbf{x})$ so that $\mathbf{x} \in V_{\alpha(x)}$. Then there are open balls $B(\mathbf{x})$ and $W(\mathbf{x})$, centered at $\mathbf{x}$, with

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset V_{\alpha(\mathbf{x})}.$$

Since K is compact, there are points $\mathbf{x}_1, \ldots, \mathbf{x}_s$ in K such that

$$K \subset B(\mathbf{x}_1) \cup \ldots \cup B(\mathbf{x}_s).$$

By Lemma 2, there are functions $b_1, \ldots, b_s \in C^\infty(\mathbb{R}^n)$ (formed by translating and scaling the basic function b, operations which preserve its smoothness), such that $b_i(\mathbf{x}) > 0$ on $B(\mathbf{x}_i)$, $b_i(\mathbf{x}) = 0$ outside $W(\mathbf{x}_i)$, and $0 \leq b_i(\mathbf{x}) \leq 1$ on $\mathbb{R}^n$.

Notice that for all $\mathbf{x} \in K$, $\sum b_j(\mathbf{x}) \neq 0$, because by the non-negativity of the b_js, equality would imply $\forall j : b_j(\mathbf{x}) = 0$, which would imply $\forall j : \mathbf{x} \notin B(\mathbf{x}_j)$. This is a contradiction, since $\{B(\mathbf{x}_j)\}$ covers K.

Let

$$\psi_i = \frac{b_i}{\sum_{j=1}^s b_j}, \quad 1 \leq i \leq s,$$

then for all $\mathbf{x} \in K$,

$$\sum_{i=1}^s \psi_i(\mathbf{x}) = \frac{\sum_{i=1}^s b_i(\mathbf{x})}{\sum_{j=1}^s b_j(\mathbf{x})} = 1,$$

so (c) is satisfied. Clearly (b) is satisfied, and (a) is satisfied, because

$$0 \leq \psi_i = \frac{b_i}{b_i + \sum_{j \neq i} b_j} \leq b_i \leq 1, \quad 1 \leq i \leq s.$$

□

Problem 1, pg 196. Define

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

Proof. Let $g(x) = e^{-1/x^2}$ so that

$$f(x) = \begin{cases} g(x) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Clearly $g'(x) = 2x^{-3}g(x)$. Assume $g^{(n)}(x) = \sum_{k=1}^{n} c_k x^{-(2k+n)} g(x)$, then

$$\begin{aligned} g^{(n+1)}(x) = \left(\sum_{k=1}^{n} c_k x^{-(2k+n)} g(x) \right)' &= \sum_{k=1}^{n} c_k \left(x^{-(2k+n)} g(x) \right)' \\ &= \sum_{k=1}^{n} c_k \left(-(2k+n) x^{-(2k+n+1)} g(x) + 2x^{-3} x^{-2k+n} g(x) \right) \\ &= \sum_{k=1}^{n} \left(\tilde{c}_{k,1} x^{-(2k+n+1)} g(x) + \tilde{c}_{k,2} x^{-(2k+n+3)} g(x) \right) \\ &= \sum_{k=1}^{n} \left(\tilde{c}_{k,1} x^{-(2k+n+1)} g(x) + \tilde{c}_{k,2} x^{-(2(k+1)+n+1)} g(x) \right) \\ &= \sum_{k=1}^{n+1} \tilde{c}_k x^{-(2k+(n+1))} g(x), \end{aligned}$$

where $\{\tilde{c}_{k,1}\}, \{\tilde{c}_{k,2}\}$, and $\{\tilde{c}_k\}$ are constants. Therefore by induction, the n-th derivative of $g(x) = e^{-1/x^2}$ can be expressed as $g^{(n)}(x) = \sum_{k=1}^{n} c_k x^{-(2k+n)} g(x)$, where the $\{c_k\}$ are constants. If $x_0 \neq 0$, then there exists a $\delta > 0$ such that if x in $N_\delta(x_0)$, $f(x) = g(x)$, therefore $f^{(n)}(x_0) = g^{(n)}(x_0)$. That is, f has derivatives of all orders at all $x \neq 0$.

Note that if $r \in Q$, then there are integers n and m such that $x^n \leq x^r \leq x^m$ for all $x > 1$. Therefore for all $x > 1$, $x^n e^{-x} \leq x^r e^{-x} \leq x^m e^{-x}$. By Theorem 8.6, $\lim_{x\to\infty} x^n e^{-x} = 0$ and $\lim_{x\to\infty} x^m e^{-x} = 0$, so $\lim_{x\to\infty} x^r e^{-x} = 0$.

Therefore if $d \in Q$, $d < 0$,

$$\lim_{h\to 0^+} h^d e^{-1/h^2} = \lim_{x\to\infty} x^{|d|/2} e^{-x} = 0,$$

and

$$\lim_{h\to 0^-} h^d e^{-1/h^2} = \lim_{x\to\infty} -x^{|d|/2} e^{-x} = 0,$$

where the substitution $x = 1/h^2$ has been made. Since

$$\lim_{h\to 0^+} h^d e^{-1/h^2} = \lim_{h\to 0^-} h^d e^{-1/h^2} = 0,$$

$\lim_{h\to 0} h^d e^{-1/h^2} = 0$.

Thus,

$$f'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} h^{-1} e^{-1/h^2} = 0.$$

Assume $f^{(n)}(0) = 0$, then

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{h\to 0} \frac{f^{(n)}(h)}{h} = \lim_{h\to 0} \frac{\sum_{k=1}^{n} c_k h^{-(2k+n)} e^{-1/h^2}}{h} \\ &= \sum_{k=1}^{n} c_k \lim_{h\to 0} h^{d_k} e^{-1/h^2} = \sum_{k=1}^{n} c_k \cdot 0 = 0 \end{aligned}$$

where $d_k = -(2k + n + 1) \in Q$.

By induction, $f^{(n)}(0) = 0$ for all positive integers n.

Since $f^{(n)}(x)$ exists for all $x \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$, $f \in C^\infty(\mathbb{R})$. $\square$