

THEOREMS FROM THE BOOK

Theorem 1. $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

PROBLEMS

Page 138, Problem 1. Suppose α increases on $[a, b]$, $a \leq x_o \leq b$, α is continuous at x_o , $f(x_o) = 1$, and $f(x) = 0$ if $x \neq x_o$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int_a^b f d\alpha = 0$.

Proof. Since α is continuous at x_o , for any $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x_o, x) < \delta \Rightarrow |\alpha(x) - \alpha(x_o)| < \epsilon/2$; by the triangle inequality, there exist points $x_l < x_r$ such that $x_l < x_o < x_r$ and $\alpha(x_r) - \alpha(x_l) < \epsilon$. Choose a partition $P = \{a = x_0 < x_1 < \dots < x_l = x_{i-1} < x_o < x_r = x_i < \dots < x_n = b\}$ of $[a, b]$. Let $M_j = \sup_{[x_{j-1}, x_j]} f(x)$ and $m_j = \inf_{[x_{j-1}, x_j]} f(x)$ for $j \in \{1, 2, \dots, n\}$. Then

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{j=1}^n (M_j - m_j) \Delta\alpha_j,$$

where $\Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1})$. Clearly for $j \neq i$, $M_j = m_j = 0$, while $M_i = 1$ and $m_i = 0$, so

$$U(P, f, \alpha) - L(P, f, \alpha) = \Delta\alpha_i = \alpha(x_r) - \alpha(x_l) < \epsilon.$$

By Theorem 1, $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Since $f \in \mathcal{R}(\alpha)$ on $[a, b]$,

$$\int_a^b f d\alpha = \int_a^b f d\alpha = \sup L(P, f, \alpha).$$

Let P' be a partition of $[a, b]$. Then

$$L(P', f, \alpha) = \sum_{j=1}^n m_j \Delta\alpha_j = 0,$$

because all $m_j = 0$; therefore $\sup L(P, f, \alpha) = 0$, or

$$\int_a^b f d\alpha = 0.$$

□

Page 138, Problem 2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof. Assume f is continuous on $[a, b]$ and $f(x_o) \neq 0$ for some $x_o \in [a, b]$. Then since f is continuous on $[a, b]$, there exists a $\delta > 0$ such that $d(x_o, x) < \delta \Rightarrow |\alpha(x) - \alpha(x_o)| < \frac{f(x_o)}{2}$; that is, $d(x_o, x) < \delta \Rightarrow \alpha(x) > f(x_o)/2$. Let $P = \{a = x_0 < x_1 < \dots < x_{i-1} < x_o < x_i < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$ such that $\max\{|x - x_i|, |x - x_{i-1}|\} < \delta$. Let $M_j = \sup_{[x_{j-1}, x_j]} f(x)$, $m_j = \inf_{[x_{j-1}, x_j]} f(x)$ and $\Delta x_j = x_j - x_{j-1}$,

for $j \in \{1, 2, \dots, n\}$. Since $f \geq 0$ on $[a, b]$, $m_j \geq 0$ for $j \neq i \in \{1, 2, \dots, n\}$; from the construction of P , $m_i \geq f(x_o)/2$. Therefore

$$L(P, f, x) = \sum_{j=1}^n m_j \Delta x_j \geq m_i \Delta x_i \geq \frac{f(x_o)}{2} (x_i - x_{i-1}) > 0.$$

If f is integrable,

$$L(P, f, x) > 0 \Rightarrow \sup L(P', f, x) = \int_a^b f dx > 0.$$

Therefore, if $f \geq 0$, f is continuous on $[a, b]$, and $f(x_o) \neq 0$ for some $x_o \in [a, b]$, and f is integrable on $[a, b]$, $\int_a^b f(x) dx > 0$. As a result, if $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$, $f(x) = 0$ for all $x \in [a, b]$. \square

Page 138, Problem 4. If $f(x) = 0$ for all irrational x , $f(x) = 1$ for all rational x , prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Let $P = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$, $M_j = \sup_{[x_{j-1}, x_j]} f(x)$, and $m_j = \inf_{[x_{j-1}, x_j]} f(x)$ for $j \in \{1, 2, \dots, n\}$. Then since it is possible to find a rational or irrational number between x_{j-1} and x_j , $m_j = 0$ and $M_j = 1$ for all $j \in \{1, 2, \dots, n\}$. Therefore

$$L(P, f, x) = \sum_{j=1}^n m_j (x_j - x_{j-1}) = 0,$$

and

$$U(P, f, x) = \sum_{j=1}^n M_j (x_j - x_{j-1}) = x_n - x_0 = b - a,$$

so for all P and $\epsilon = \frac{1}{2}(b - a)$,

$$U(P, f, x) - L(P, f, x) = b - a > \epsilon.$$

By Theorem 1, $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$. \square

Extra Problem. Let

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases} \quad \varphi(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ \frac{1}{n}, & x = \frac{m}{n} \end{cases},$$

where $m, n \in \mathbb{Z}$, m and n have no common factors and $n \neq 0$, is $f(\varphi(x))$ integrable?

If $x \in \mathbb{Q}$, $\varphi(x) \neq 0$, so $f(\varphi(x)) = 1$, and if $x \notin \mathbb{Q}$, $\varphi(x) = 0$, so $f(\varphi(x)) = 0$; that is,

$$f(\varphi(x)) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}.$$

By problem 4, $f(\varphi(x))$ is not integrable on any interval $[a, b]$.