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Theorems from the book

Theorem 1. *$f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that*

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon. \tag{1}$$

Theorem 2. (**a**) *If (1) holds for some P and some ϵ, then (1) holds (with the same ϵ) for every refinement of P.*

(**b**) *If (1) holds for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon.$$

(**c**) *If $f \in \mathcal{R}(\alpha)$ and the hypotheses of (b) hold, then*

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \epsilon.$$

Theorem 3. *Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.*

Problems

Page 138, Problem 3. Define three functions β_1,β_2,β_3 as follows:, $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0^+) = f(0)$ and that then

$$\int f d\beta_1 = f(0).$$

(b) State and prove a similar result for β_2.

(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

Proof.

(**a**) Assume $f(0^+) = 0$. Let $\epsilon > 0$; then there exists a $\delta > 0$ such that $|f(0) - f(x)| < \epsilon/2$ for all x such that $0 \le x < \delta$. Let $P = \{-1 = x_0 < \cdots < x_{i-1} = 0 < x_i = \delta < \cdots < x_n = 1\}$ be a partition of $[-1, 1]$. Then for $j \ne i$, $\Delta{\beta_1}_j = 0$, and $\Delta{\beta_1}_i = 1$, so

$$U(P,f,\beta_1) - L(P,f,\beta_1) = M_i - m_i < \left(f(0) + \frac{\epsilon}{2}\right) - \left(f(0) - \frac{\epsilon}{2}\right) = \epsilon,$$

or by Theorem 1, $f \in \mathcal{R}(\beta_1)$. Let $P = \{-1 = x_0 < \cdots < x_{i-1} \leq 0 \leq x_i < \cdots < x_n = 1\}$ be a partition of $[-1, 1]$, then $0 \in [x_{i-1}, x_i]$, so

$$L(P, f, \beta_1) = m_i \leq f(0),$$

or $f(0)$ is an upper bound for the set $\{L(P', f, \beta_1) \mid P' = \{-1 = x_0 < \cdots < x_n = 1\}\}$. Assume l is a upper bound for the same set, such that $l < f(0)$. Let $d = f(0) - l$. Then since $f(0+) = f(0)$, there exists a δ such that $0 \leq x < \delta \Rightarrow f(0) - f(x) < d = f(0) - l \Rightarrow f(x) > l$. Choose a partition $P = \{-1 = x_0 < \cdots < x_{i-1} = 0 < x_i = \delta < \cdots < x_n = 1\}$; then $L(P, f, \beta_1) = m_i > l$. This contradiction shows that $f(0)$ is the least upper bound of the given set, or

$$f(0) = \sup L(P', f, \beta_1) = \int f d\beta_1.$$

Assume $f \in \mathcal{R}(\beta_1)$ and let $\epsilon > 0$, then by Theorem 1, there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Let P' be a refinement of P such that $P' = \{-1 = x_0 < \cdots < x_{i-1} = 0 < x_i = \delta < \cdots < x_n = 1\}$; then by Theorem 2,

$$\sum_{j=1}^{n} |f(s_j) - f(t_j)| \Delta\beta_{1j} = |f(s_i) - f(t_i)| = |f(0) - f(t_i)| < \epsilon,$$

for all $t_i \in [0, \delta]$ (where $s_i \equiv 0$). Therefore for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < x < \delta \Rightarrow |f(0) - f(x)| < \epsilon$; that is, $f(0+) = f(0)$.

These arguments show that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$, and that then $\int f d\beta_1 = f(0)$.

(b) Statement: $f \in \mathcal{R}(\beta_2)$ if and only if $f(0-) = f(0)$, and then

$$\int f d\beta_2 = f(0).$$

Proof. Assume $f(0^-) = 0$. Let $\epsilon > 0$; then there exists a $\delta > 0$ such that $|f(0) - f(x)| < \epsilon/2$ for all x such that $-\delta < x \leq 0$. Let $P = \{-1 = x_0 < \cdots < x_{i-1} = -\delta < x_i = 0 < \cdots < x_n = 1\}$ be a partition of $[-1, 1]$. Then for $j \neq i$, $\Delta\beta_{2j} = 0$, and $\Delta\beta_{2i} = 1$, so

$$U(P, f, \beta_2) - L(P, f, \beta_2) = M_i - m_i < \left(f(0) + \frac{\epsilon}{2}\right) - \left(f(0) - \frac{\epsilon}{2}\right) = \epsilon,$$

or by Theorem 1, $f \in \mathcal{R}(\beta_2)$. Let $P = \{-1 = x_0 < \cdots < x_{i-1} \leq 0 \leq x_i < \cdots < x_n = 1\}$ be a partition of $[-1, 1]$, then $0 \in [x_{i-1}, x_i]$, so

$$L(P, f, \beta_2) = m_i \leq f(0),$$

or $f(0)$ is an upper bound for the set $\{L(P', f, \beta_2) \mid P' = \{-1 = x_0 < \cdots < x_n = 1\}\}$. Assume l is a upper bound for the same set, such that $l < f(0)$. Let $d = f(0) - l$. Then since $f(0-) = f(0)$, there exists a δ such that $-\delta < x \leq 0 \Rightarrow f(0) - f(x) < d = f(0) - l \Rightarrow f(x) > l$. Choose a partition $P = \{-1 = x_0 < \cdots < x_{i-1} = -\delta < x_i = 0 < \cdots < x_n = 1\}$; then $L(P, f, \beta_2) = m_i > l$. This contradiction shows that $f(0)$ is the least upper bound of the given set, or

$$f(0) = \sup L(P', f, \beta_2) = \int f d\beta_2.$$

Assume $f \in \mathcal{R}(\beta_2)$ and let $\epsilon > 0$, then by Theorem 1, there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_2) - L(P, f, \beta_2) < \epsilon$. Let P' be a refinement of P such that $P' = \{-1 = x_0 < \cdots < x_{i-1} = -\delta < x_i = 0 < \cdots < x_n = 1\}$; then by Theorem 2,

$$\sum_{j=1}^{n} |f(s_j) - f(t_j)| \Delta\beta_{2j} = |f(s_i) - f(t_i)| = |f(0) - f(t_i)| < \epsilon,$$

for all $t_i \in [-\delta, 0]$ (where $s_i \equiv 0$). Therefore for every $\epsilon > 0$, there exists a $\delta > 0$ such that $-\delta < x < 0 \Rightarrow |f(0) - f(x)| < \epsilon$; that is, $f(0-) = f(0)$.

(**c**) Assume f is continuous at 0 and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that $|f(0) - f(x)| < \epsilon/2$ for all $x \in (-\delta, \delta)$. Let $P = \{-1 = x_0 < \cdots < x_{i-1} = -\delta < x_i = \delta < \cdots < x_n = 1\}$. Then $\Delta\beta_{3j} = 0$ for $j \neq i$, and $\Delta\beta_{3i} = 1$, so

$$U(P, f, \beta_3) - L(P, f, \beta_3) = M_i - m_i < \left(f(0) + \frac{\epsilon}{2}\right) - \left(f(0) - \frac{\epsilon}{2}\right) = \epsilon,$$

or by Theorem 1, $f \in \mathcal{R}(\beta_3)$.

Assume $f \in \mathcal{R}(\beta_3)$ and let $\epsilon > 0$, then by Theorem 1, there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon/2$. Let P' be a refinement of P such that $P' = \{-1 = x_0 < \cdots < x_{i-2} = -\delta < x_{i-1} = 0 < x_i = \delta < \cdots < x_n = 1\}$; then by Theorem 2,

$$\sum_{j=1}^{n} |f(s_j) - f(t_j)|\Delta\beta_{3j} = \frac{|f(s_{i-1}) - f(t_{i-1})| + |f(s_i) - f(t_i)|}{2} = \frac{|f(0) - f(t_{i-1})| + |f(0) - f(t_i)|}{2} < \frac{\epsilon}{2},$$

or

$$|f(0) - f(t_{i-1})| + |f(0) - f(t_i)| < \epsilon$$

for all $t_{i-1} \in [-\delta, 0]$ and all $t_i \in [0, \delta]$ (where $s_{i-1} \equiv s_i \equiv 0$). Since both terms are positive, for every $\epsilon > 0$, there exists a δ such that $|f(0) - f(x)| < \epsilon$ for all x such that $|x| < \delta$; that is, f is continuous at 0.

Therefore $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

(**d**) Assume f is continuous at 0. By (c), $f \in \mathcal{R}(\beta_3)$. Let $P = \{-1 = x_0 < \cdots < x_{i-1} < 0 < x_i < \cdots < x_n = 1\}$ be a partition of $[-1, 1]$, then $0 \in [x_{i-1}, x_i]$, so

$$L(P, f, \beta_3) = m_i \leq f(0),$$

or $f(0)$ is an upper bound for the set $\{L(P', f, \beta_3) \mid P' = \{-1 = x_0 < \cdots < x_n = 1\}\}$. Assume l is a upper bound for the same set, such that $l < f(0)$. Let $d = f(0) - l$. Then since f is continuous at 0, there exists a δ such that $|x| < \delta \Rightarrow f(0) - f(x) < d = f(0) - l \Rightarrow f(x) > l$. Choose a partition $P = \{-1 = x_0 < \cdots < x_{i-1} = -\delta < x_i = \delta < \cdots < x_n = 1\}$; then $L(P, f, \beta_3) = m_i > l$. This contradiction shows that $f(0)$ is the least upper bound of the given set, or

$$f(0) = \sup L(P', f, \beta_3) = \int f d\beta_3.$$

Furthermore, since f is continuous at 0, $f(0+) = f(0-) = f(0)$, so by (a), (b), and the above argument,

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

□

Page 138, Problem 5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume $f^3 \in \mathcal{R}$?

Let $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \notin Q$, then f is bounded and $f^2 = 1 \in \mathcal{R}$ on $[a, b]$. However, let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ and $\epsilon_0 = b - a$. Then

$$U(P, f, x) - L(P, f, x) = \sum_{j=1}^{n}(M_j - m_j)\Delta x_j = \sum_{j=1}^{n} 2\Delta x_j = 2(b - a) > \epsilon_0,$$

so by Theorem 1, $f \notin \mathcal{R}$. Therefore the answer to the first question is no.

Assume f is bounded on $[a, b]$ and $f^3 \in \mathcal{R}$. Let $M > 0$ be such that $f(x) \in [-M, M]$ for all $x \in [a, b]$; then $f^3(x) \in [-M^3, M^3]$. Let $\phi(x) = \sqrt[3]{x}$; clearly ϕ is continuous on $[-M^3, M^3]$. By Theorem 3, $h = \phi(f^3) = f$ is in $\mathcal{R}$ on $[a, b]$. Therefore the answer changes (to yes) if we assume $f^3 \in \mathcal{R}$.