Alex Gittens
October 29, 2004

THEOREMS FROM THE BOOK

Theorem 1 (4.9)**.** *Let f and g be complex continuous functions on a metric space X. Then $f+g$, fg, and f/g are continuous on X, provided in the last case that $g(x) \neq 0$ for all $x \in X$.*

Theorem 2 (4.19)**.** *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

PROBLEMS

Page 100, Problem 18. Every rational x can be written in the form $x = \frac{m}{n}$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0, & (x \text{ irrational}) \\ \frac{1}{n}, & x = \frac{m}{n} \end{cases}.$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof. Let x be in $\mathbb{R}$ and $\{x_n\} \subset \mathbb{R}$ be such that $x_n \to x$. Then there are two cases: $\{x_n\}$ contains a finite number of rationals $\{x_{n_k}\}_{k=1}^N$, and $\{x_n\}$ contains a subsequence of rationals $\{x_{n_k}\}$. In the first case, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_{n+n_N}) = 0$, since f is 0 at every irrational point.

In the second case, let k_1 be the denominator of x_{n_1}. There are only finitely many rationals with denominators less than or equal to k_1, so there is a rational of that form, r_1, closest to x. Define $d_1 = |x - r_1|$. Since $x_{n_k} \to x$, there is a rational closer than d_1 to x; denote the denominator of this rational by k_2; note that $k_2 > k_1$. Proceeding in this manner, define a strictly increasing sequence $\{k_n\}$. It is clear that $k_n \to \infty$. Therefore $\lim_{n\to\infty} f(x_n) = \lim_{n_k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} \frac{1}{k_n} = 0$.

If x is irrational, this shows that $x_n \to x \Rightarrow f(x_n) \to f(x)$, or that f is continuous at x. If x is rational, this shows that if $x_n^l \to x$ and $x_n^r \to x$ such that for all n, $x_n^l < x$ and $x_n^r > x$, then $f(x_n^l) \to 0 \neq f(x)$ and $f(x_n^r) \to 0 \neq f(x)$. That is, f has a discontinuity of the first kind at x. ☐

Extra Problem I. If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x)$ exists, prove f is uniformly continuous.

Proof. Let $\epsilon > 0$ and $\lim_{x\to\infty} f(x) = C$, then there exists a $c \in \mathbb{R}$ such that $x > c \Rightarrow |f(x) - C| < \frac{\epsilon}{2}$. By the triangle inequality, $x, y > c \Rightarrow |f(x) - f(y)| < \epsilon$. Therefore for all $\delta > 0$, $x, y > c$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

By Theorem 2, f is uniformly continuous on $[0, c]$; therefore, there is a δ_0 such that $x, y \in [0, c]$ and $|x - y| < \delta_0 \Rightarrow |f(x) - f(y)| < \epsilon$.

Since f is continuous at c, there is a δ_c such that $|c - x| < \delta_c \Rightarrow |f(c) - f(x)| < \frac{\epsilon}{2}$. Again by the triangle inequality, this shows that when $x, y \in N_{\delta_c}(c)$, $|f(x) - f(y)| < \epsilon$.

Take $\delta = \min(2\delta_c, \delta_0)$, then $x, y \in [0, \infty)$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. That is, f is uniformly continuous on $[0, \infty)$. ☐

Extra Problem II. Prove $f:\ x \mapsto \frac{1}{x}$ is continuous on $(0,1)$ but is not uniformly continuous on $(0,1)$.

Proof. The function $f:\ x \mapsto x$ is clearly continuous on $(0,1)$, and $\forall x \in (0,1), f(x) \neq 0$. The function $g:\ x \mapsto 1$ is also continuous on $(0,1)$. By Theorem 1, the function $h: x \mapsto \frac{1}{x}$ is continuous on $(0,1)$.

Take $\epsilon_0 = 1$. Let $x = \frac{1}{n}$, $y = \frac{1}{n+1}$, where $n \in \mathbb{Z}^+$. Then

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(1+n)}$$

and $|f(x) - f(y)| = 1$. Since for all $\delta > 0$ it is possible to choose $n \in \mathbb{Z}^+$ so that $\frac{1}{n(1+n)} < \frac{1}{n} < \delta$, there is an $\epsilon > 0$ (namely $\epsilon_0 = 1$) such that for all $\delta > 0$, there are $x, y \in (0,1)$ such that $|x-y| < \delta$ and $|f(x)-f(y)| \geq \epsilon$. Therefore $f: x \mapsto \frac{1}{x}$ is not uniformly continuous on $(0,1)$. ☐

Extra Problem III. Use the $\epsilon - \delta$ definition to prove $f:\ x \mapsto \sqrt[3]{x} + \sin x$ is continuous on $(-\infty, \infty)$.

Proof. Consider two numbers a, b such that $a > b \geq 0$ or $b < a \leq 0$, then $a^{\frac{1}{3}} > b^{\frac{1}{3}} \Rightarrow a^{\frac{2}{3}} \geq a^{\frac{1}{3}} b^{\frac{1}{3}} \Rightarrow a^{\frac{2}{3}} b^{\frac{1}{3}} \geq a^{\frac{1}{3}} b^{\frac{2}{3}} \Rightarrow$

$$(a^{1/3} - b^{1/3})^3 = a - 3\,a^{\frac{2}{3}}\,b^{\frac{1}{3}} + 3\,a^{\frac{1}{3}}\,b^{\frac{2}{3}} - b \leq a - b = ((a-b)^{\frac{1}{3}})^3,$$

or $(a^{\frac{1}{3}} - b^{\frac{1}{3}}) \leq (a-b)^{\frac{1}{3}} \Rightarrow |a^{\frac{1}{3}} - b^{\frac{1}{3}}| \leq |(a-b)^{\frac{1}{3}}|$. This shows that

$$|a^{\frac{1}{3}} - b^{\frac{1}{3}}| \leq |a-b|^{\frac{1}{3}},$$

for a, b either both nonnegative or both nonpositive.

Let $\epsilon > 0$ and consider $x \in \mathbb{R}$. If y has the same sign as x (or any sign if $x = 0$) and $|y - x| < \epsilon^3$, then

$$|\sqrt[3]{x} - \sqrt[3]{y}| \leq |x-y|^{1/3} < \epsilon,$$

Take

$$\delta_1 = \begin{cases} \min(\epsilon^3, |x|),\ x \neq 0 \\ \epsilon^3,\ x = 0 \end{cases},$$

then $|x - y| < \delta_1 \Rightarrow |\sqrt[3]{x} - \sqrt[3]{y}| < \epsilon$.

Furthermore, for all x, y such that $|x - y| < \epsilon$,

$$|\sin x - \sin y| < |x - y| < \epsilon.$$

Take $\delta = \min(\epsilon, \delta_1)$, then

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq |\sqrt[3]{x} - \sqrt[3]{y}| + |\sin x - \sin y| < 2\epsilon.$$

Replacing 2ϵ with ϵ in the above inequality shows that for all $\epsilon > 0$ and $x \in \mathbb{R}$, there exists a δ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$; that is, f is continuous on R. ☐