

THEOREMS FROM THE BOOK

Theorem 1 (5.3). *Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg , and f/g are differentiable at x , and*

- (a) $(f + g)'(x) = f'(x) + g'(x)$;
- (b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
- (c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

In (c), we assume of course that $g(x) \neq 0$.

Theorem 2 (Chain Rule, 5.5). *Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f , and g is differentiable at the point $f(x)$. If*

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x , and

$$h'(x) = g'(f(x))f'(x).$$

Theorem 3 (Mean Value Theorem, 5.10). *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b) , then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Theorem 4 (5.11). *Suppose f is differentiable in (a, b) .*

- (a) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*
- (b) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*
- (c) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

PROBLEMS

Page 114, Problem 1. Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is constant.

Proof. Let $y, x \in \mathbb{R}$, then

$$\begin{aligned} |f(x) - f(y)| \leq (x - y)^2 &\Rightarrow -(x - y)^2 \leq f(x) - f(y) \leq (x - y)^2 \\ &\Rightarrow y - x \leq \frac{f(x) - f(y)}{x - y} \leq x - y, \end{aligned}$$

then as $y \rightarrow x$, $\frac{f(x)-f(y)}{x-y} \rightarrow 0$, so $f'(x) = 0$ for all $x > 0$. By Theorem 4, f is constant on all intervals of the form (a, b) . Therefore $f(x) = f(y)$ for all real x, y — because if this was not so, there would be numbers $y > x$ such that $f(y) \neq f(x)$, that is, f would not be constant on the interval (x, y) . Therefore, f is constant. \square

Page 114, Problem 5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Proof. For all $x > 0$, by Theorem 3, there exists a $y \in (x, x+1)$ satisfying

$$g(x) = f(x+1) - f(x) = (x+1-x)f'(y) = f'(y).$$

As $x \rightarrow \infty$, $y \rightarrow \infty$, therefore $g(x) = f'(y) \rightarrow 0$ as $x \rightarrow \infty$. \square

Page 114, Problem 6. Suppose

- (a) f is continuous for $x \geq 0$,
- (b) $f'(x)$ exists for $x > 0$,
- (c) $f(0) = 0$,
- (d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Proof. By Theorem 3, for each $x > 0$, there is a $y \in (0, x)$ such that $f(x) - f(0) = f'(y)(x - 0)$, or $f(x) = f'(y)x$. By Theorem 1,

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - f'(y)}{x}.$$

Since f' is monotonically increasing, $x > y$, and $x > 0$, $g'(x) \geq 0$. By Theorem 4, this shows g is monotonically increasing. \square

Page 115, Problem 12. If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x , and show that $f^{(3)}(0)$ does not exist.

Let $a(x) = |x|$ denote the absolute value function, and $c(x) = x^3$ denote the cubing function.

Then

$$c'(x) = \lim_{t \rightarrow x} \frac{c(t) - c(x)}{t - x} = \lim_{t \rightarrow x} \frac{t^3 - x^3}{t - x} = \lim_{t \rightarrow x} \frac{(t-x)(t^2 + tx + x^2)}{t - x} = \lim_{t \rightarrow x} t^2 + tx + x^2 = 3x^2,$$

and for $x \neq 0$,

$$a'(x) = \lim_{t \rightarrow x} \frac{a(t) - a(x)}{t - x} = \lim_{t \rightarrow x} \frac{|t| - |x|}{t - x} = \lim_{t \rightarrow x} \frac{t \operatorname{sgn} t - x \operatorname{sgn} x}{t - x} = \operatorname{sgn} t \lim_{t \rightarrow x} \frac{t - x}{t - x} = \operatorname{sgn} x.$$

By Theorem 2,

$$f'(x) = c'(a(x))a'(x)$$

when $x \neq 0$. When $x = 0$,

$$f'(x) = \lim_{t \rightarrow 0} \frac{|t|^3 - |0|^3}{t - 0} = \lim_{t \rightarrow 0} |t|^2 \operatorname{sgn} t = 0.$$

Therefore

$$f'(x) = \begin{cases} 3|x|^2 \operatorname{sgn} x = 3x|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

By Theorem 1, since x and $|x|$ are differentiable when $x \neq 0$, $f''(x) = 3x \operatorname{sgn} x + 3|x|$ when $x \neq 0$. When $x = 0$,

$$f''(0) = \lim_{t \rightarrow 0} \frac{3t|t|}{t} = \lim_{t \rightarrow 0} 3|t| = 0.$$

Therefore

$$f''(x) = \begin{cases} 3x \operatorname{sgn} x + 3|x| = 6|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Let $t_n^- = -\delta/n$ and $t_n^+ = \delta/n$ for a $\delta > 0$. Then clearly $t_n^- \rightarrow 0$ and $t_n^+ \rightarrow 0$ as $n \rightarrow \infty$, yet $\frac{f''(t_n^-)}{t_n^-} \rightarrow -1$ and $\frac{f''(t_n^+)}{t_n^+} \rightarrow 1$, therefore $f^{(3)}(0) = \lim_{t \rightarrow 0} \frac{f''(t)}{t}$ does not exist.