Alex Gittens
Matrix Theory

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HOMEWORK 4

Problem 1. Let $A \in M_n$ be such that $\langle AX, X\rangle \geq 0$ for all $X \in \mathbf{C}^n$. Prove that $a_{ii} \geq 0$ for $1 \leq i \leq n$ where $A = [a_{ij}]$.

Let $e_i = (0, 0, \dots, 1, 0, \dots, 0)^t$ be the i-th standard basis vector of $\mathbf{C}^n$. Then $\langle Ae_i, e_i\rangle \geq 0$ by the positivity of A, and $\langle Ae_i, e_i\rangle = e_i A e_i = a_{ii}$, so for $1 \leq i \leq n$, $a_{ii} \geq 0$.

Problem 2. Let $A \in M_n$, $A = [a_{ij}]$ satisfy $\langle AX, X\rangle > 0$ for all $X \in \mathbf{C}^n$. Let $a = \max_{i,j} |a_{ij}|$. Prove $a = a_{ii}$ for some i.

Proof. Let A be as given, and let e_i be the i-th standard basis vector of $\mathbf{C}^n$. By problem (3) below, there is a positive B such that $A = B^2$, so

$$|a_{ij}| = |\langle Ae_j, e_i\rangle| = |\langle Be_j, B^\star e_i\rangle| = |\langle Be_j, Be_i\rangle| \leq \|Be_j\|\|Be_i\| \leq \max\{\|Be_j\|^2, \|Be_i\|^2\}.$$

Assume $\|Be_j\| \geq \|Be_i\|$, then

$$|a_{ij}| \leq \|Be_j\|^2 = \langle Be_j, Be_j\rangle = \langle B^\star Be_j, e_j\rangle = \langle B^2 e_j, e_j\rangle = \langle Ae_j, e_j\rangle = a_{jj},$$

so every $|a_{ij}|$ is dominated by either a_{ii} or a_{jj}. Therefore $a = a_{ii}$ for some i. □

Problem 3. Let A be as in (1) above. Show that $A = B^2$ for some $B \in M_n$ satisfying $\langle BX, X\rangle \geq 0$ for all $X \in \mathbf{C}^n$.

Proof. Let A be as given. A is normal, so $A = U^\star DU$ for some unitary matrix U and diagonal matrix D, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Let λ_i denote the i-th eigenvalue of A, counting multiplicities; since A is positive, we know $\sigma(A) \subset \mathbf{R}^+ \cup \{0\}$. Therefore $\sqrt{\lambda_i} \geq 0$ for all i; let $D' = \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Let $X = (x_1, \dots, x_n)^t \in \mathbf{C}^n$, then

$$\langle D'X, X\rangle = \sum_{i=1}^n \sqrt{\lambda_i}|x_i|^2 \geq 0,$$

so D' is positive. Let $B = U^\star D'U$, then B is positive since matrices unitarily equivalent to positive matrices are positive. Furthermore,

$$B^2 = U^\star D'UU^\star D'U = U^\star \left(D'\right)^2 U = U^\star DU = A.$$

□

Problem 4. Find the Cholesky decomposition of

$$A = \begin{bmatrix} 4 & 2i & -i \\ -2i & 10 & 1 \\ i & 1 & 9 \end{bmatrix}$$

Problem 5. (Exercise 2, pg. 181) If $A \in M_n$ is Hermitian, show that the following three optimization problems all have the same solution:

(a) $\displaystyle\max_{x^\star x=1} x^\star Ax$

(b) $\displaystyle\max_{x\neq 0} \frac{x^\star Ax}{x^\star x}$

(c) $\displaystyle\max_{x^\star Ax} \frac{1}{x^\star x}$ when at least one eigenvalue of A is positive.

Problem 6. Let $A \in M_n$ have eigenvalues $\{\lambda_i\}$. Show that, even if A is not Hermitian, one has the bounds

$$\min_{x\neq 0}\left|\frac{x^\star Ax}{x^\star x}\right| \leq |\lambda_i| \leq \max_{x\neq 0}\left|\frac{x^\star Ax}{x^\star x}\right|, \quad i = 1, 2, \ldots, n \tag{0-1}$$

Hint: Consider $x =$ an eigenvector of A, and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to show that neither bound need be sharp.