

HOMEWORK 5

Problem 1. (Exercise 1, pg. 212) Suppose $A \in M_n$ is symmetric and $A = B + iC$ with $B, C \in M_n$ both real. Show that A is normal if and only if B and C commute. Show that A is normal if and only if $A\bar{A}$ is real. Show that A is normal if and only if A and \bar{A} commute. Give an example of a symmetric matrix that is not normal.

Problem 2. (Exercise 12, pg. 215) If $A \in M_n$ is a complex symmetric matrix and if $x, y \in \mathbf{C}^n$ are eigenvectors of A corresponding to distinct eigenvalues of A , show that $x^t y = 0$. Does this mean that x and y are orthogonal? *Hint:* Consider $x^t(Ay) = (Ax)^t y$.

Let A be as given, and x, y be eigenvectors of A corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Then

$$\lambda_1 x^t y = (Ax)^t y = x^t A^t y = x^t A y = \lambda_2 x^t y,$$

so since $\lambda_1 \neq \lambda_2$, $x^t y = 0$. In general, this does not mean x and y are orthogonal, because $x^t y = 0 \not\Rightarrow x^* y = 0$.

Problem 3. (Exercise 14, pg. 215) If $A \in M_n$ is symmetric and nonsingular, show that A^{-1} is symmetric.

Since A is symmetric and nonsingular,

$$AA^{-1} = I \Rightarrow (A^{-1})^t A^t = I \Rightarrow (A^{-1})^t A = I,$$

so $A^{-1} = (A^{-1})^t$ by the uniqueness of inverses; that is, A^{-1} is symmetric.

Problem 4. (Exercise 15, pg. 215) A real symmetric matrix is Hermitian and therefore is diagonalizable. Show that a complex symmetric matrix need not be diagonalizable. *Hint:* Consider $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ and compute A^2 .

Problem 5. (Exercise 20, pg. 243) If $A = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} \in M_n$ with $B \in M_k$, $1 \leq k < n$, show that A is normal if and only if B is normal and $C = 0$. *Hint:* Compute AA^* and A^*A . If $C^*C = 0$, then $(Cx)^*(Cx) = 0$ for all $x \in \mathbf{C}^{n-k}$ and hence $Cx = 0$ for all $x \in \mathbf{C}^{n-k}$.

Problem 6. (Exercise 1, pg. 328) Let $G(\cdot)$ be a vector norm on M_n and let $y \in \mathbf{C}^n$ be a given nonzero vector. Show that the function

$$\|x\| \equiv G(xy^*)$$

is a vector norm on \mathbf{C}^n . What is this when

$$y = [1, 1, \dots, 1]^t \text{ or } y = [1, 0, 0, \dots, 0]^t$$

Let $\|\cdot\|$ be defined as above. Since $G : M_n \rightarrow [0, \infty)$, $\|\cdot\| : M_n \rightarrow [0, \infty)$ also. If $x = 0$, then $\|x\| = G(0) = 0$, and $\|x\| = 0 \Rightarrow G(xy^*) = 0 \Rightarrow xy^* = 0 \Rightarrow x_i y^* = 0$ for $1 \leq i \leq n$. Since $y^* \neq 0$, this implies $x = 0$. Let $\alpha \in \mathbf{C}$, $x \in \mathbf{C}^n$, then $\|\alpha x\| = G(\alpha xy^*) = |\alpha|G(xy^*) = |\alpha|\|x\|$. If $x, z \in \mathbf{C}^n$, $\|x + z\| = G((x + z)y^*) \leq G(xy^*) + G(zy^*) = \|x\| + \|z\|$. Therefore $\|\cdot\|$ is a vector norm on \mathbf{C}^n .

When $y = [1, 1, \dots, 1]^t$, $\|x\| = G \left(\begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \cdots & \vdots \\ x_n & x_n & \cdots & x_n \end{bmatrix} \right)$, and when $y = [1, 0, 0, \dots, 0]^t$,

$$\|x\| = G \left(\begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right).$$

Problem 7. (Exercise 4, pg. 329) Let $G(\cdot)$ be a given vector norm on M_n , and define the function $G' : M_n \rightarrow \mathbf{R}$ by

$$G'(B) \equiv \max_{G(A)=1} G(BA)$$

Show that $G'(\cdot)$ is always a matrix norm on M_n . Show that $G'(I) = 1$ always. If $G(I) = 1$, show that $G'(B) \geq G(B)$ for all $B \in M_n$.

Let G, G' be as above. Since $G'(B)$ is the maximum of a set of nonnegative values for each $B \in M_n$, $G' : M_n \rightarrow [0, \infty)$; clearly $G'(0) = 0$. Let $B \in M_n$ be such that $G'(B) = 0$. Assume $B \neq 0$, then $B^* \neq 0$ and $G\left(\frac{B^*}{G(B^*)}\right) = 1$, so

$$G\left(B\frac{B^*}{G(B^*)}\right) = 0 \Rightarrow G(BB^*) = 0 \Rightarrow BB^* = 0 \Rightarrow \text{tr}(BB^*) = \sum_{i,j=1}^n |b_{ij}|^2 = 0,$$

so $B = 0$. This contradiction proves $G'(B) = 0 \Rightarrow B = 0$. Let $\alpha \in \mathbf{C}$ and $B \in M_n$, then

$$G'(\alpha B) = \max_{G(A)=1} G(\alpha BA) = \max_{G(A)=1} |\alpha|G(BA) = |\alpha|G'(B)$$

Let $C, B \in M_n$, then

$$\begin{aligned} G'(C + B) &= \max_{G(A)=1} G((C + B)A) \leq \max_{G(A)=1} G(CA) + G(BA) \\ &\leq \max_{G(A)=1} G(CA) + \max_{G(A)=1} G(BA) = G'(C) + G'(B), \end{aligned}$$

so G' satisfies the triangle inequality.

To verify submultiplicativity, let $B, C \in M_n$, then note $G'(BC) = 0 = G'(B)G'(C)$ when $C = 0$, so submultiplicativity is trivially satisfied in this case. Assume $C \neq 0$, then

$$\begin{aligned} G'(BC) &= \max_{G(A)=1} G(BCA) = \max_{A \neq 0} \frac{G(BCA)}{G(A)} \\ &= \max_{\substack{A \neq 0 \\ CA \neq 0}} \frac{G(BCA)}{G(CA)} \frac{G(CA)}{G(A)} \leq \max_{X \neq 0} \frac{G(BX)}{G(X)} \max_{A \neq 0} \frac{G(CA)}{G(A)} \\ &= G'(B)G'(C). \end{aligned}$$

Therefore G' is a matrix norm on M_n .

Applying the definition, $G'(I) = \max_{G(A)=1} G(IA) = 1$. Furthermore, if $G(I) = 1$, then $G'(B) = \max_{G(A)=1} G(BA) \geq G(BI) = G(B)$ for all $B \in M_n$.

Problem 8. (Exercise 5, pg. 329) If $G(\cdot)$ is a matrix norm on M_n , show that $G'(B) \leq G(B)$ for all $B \in M_n$ and if $G(I) = 1$, then $G'(\cdot) = G(\cdot)$.

If G is a matrix norm on M_n , then for every $A, B \in M_n$, $G(BA) \leq G(B)G(A)$, so

$$G'(B) = \max_{G(A)=1} G(BA) \leq \max_{G(A)=1} G(B)G(A) = G(B).$$

If additionally $G(I) = 1$, then from the above exercise, $G(B) \leq G'(B)$, so $G(B) = G'(B)$ for all $B \in M_n$.