

Exercise 2, pg. 88 (in the new edition). If ν is a signed measure, E is ν -null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. Let (X, \mathcal{M}, ν) be a measure space, with ν a signed measure. Let $E \in \mathcal{M}$ and assume $|\nu|(E) = 0$, then $\nu^+(E) + \nu^-(E) = 0$ and $0 \leq \nu^+(E), 0 \leq \nu^-(E)$ implies $\nu(E) = \nu^+(E) + \nu^-(E) = 0 + 0 = 0$, so if $S \subseteq E$, $S \in \mathcal{M}$, then $|\nu|(S) \leq |\nu|(E) = 0$ so $\nu(S) = 0$. Therefore E is ν -null. Assume E is ν -null, and let $X = P \dot{\cup} N$ be a Hahn decomposition for ν , then

$$\begin{aligned} |\nu|(E) &= \nu^+(E) + \nu^-(E) = \nu^+(E \cap P) + \nu^+(E \cap N) + \nu^-(E \cap P) + \nu^-(E \cap N) \\ &= \nu^+(E \cap P) + \nu^-(E \cap N) = \nu(E \cap P) + \nu(E \cap N) = 0, \end{aligned}$$

so $|\nu|(E) = 0$. Therefore E is ν -null iff $|\nu|(E) = 0$.

Let μ be another signed measure on (X, \mathcal{M}) .

- (i) Assume $\nu \perp \mu$, then there exist $E, F \in \mathcal{M}$ such that $E \dot{\cup} F = X$, ν is null on E , and μ is null on F . By the above result, $|\nu|(E) = 0$, and for all $S \subseteq E$, $S \in \mathcal{M}$, $|\nu|(S) \leq |\nu|(E) = 0$, so $|\nu|(S) = 0$. Therefore E is $|\nu|$ -null, so $|\nu| \perp \mu$.
- (ii) Assume $|\nu| \perp \mu$, then there exist $E, F \in \mathcal{M}$ such that $|\nu|$ is null on E and μ is null on F , and $E \dot{\cup} F = X$. Let $S \subseteq E$, $S \in \mathcal{M}$, then $\nu^+(S) \leq |\nu|(S) = 0$ and $\nu^-(S) \leq |\nu|(S) = 0$, so ν^+ and ν^- are null on E . Therefore $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.
- (iii) Assume $\nu^+ \perp \mu$ and $\nu^- \perp \mu$, then by (i), $|\mu| \perp \nu^+$ and $|\mu| \perp \nu^-$. Therefore there are sets E^+, E^-, F^+, F^- such that $E^+ \dot{\cup} F^+ = X = E^- \dot{\cup} F^-$ and ν^+ is null on E^+ , ν^- is null on E^- , and $|\mu|$ is null on F^+ and F^- . Let $F = F^+ \cup F^-$ and $E = F^c = E^+ \cap E^-$. Then $|\mu|(F) \leq |\mu|(F^+) + |\mu|(F^-) = 0$ so by the first result shown above, F is μ -null. Furthermore, let $S \subseteq E$, $S \in \mathcal{M}$, then $\nu^+(S) \leq \nu^+(E) \leq \nu^+(E^+) = 0$ and $\nu^-(S) \leq \nu^-(E) \leq \nu^-(E^-) = 0$, so $|\nu|(S) = 0$. Therefore $\mu \perp |\nu|$.
- (iv) Assume $|\nu| \perp \mu$, then there exist $E, F \in \mathcal{M}$ such that $E \dot{\cup} F = X$ and $|\nu|$ is null on E and μ is null on F . Then $|\nu|(E) = 0$ so by the first result shown above, E is ν -null. Therefore $\nu \perp \mu$.

Together, (i)-(iv) show $\nu \perp \mu \Leftrightarrow |\nu| \perp \mu \Leftrightarrow \nu^+ \perp \mu$ and $\nu^- \perp \mu$. □

Exercise 3, pg. 88 (in the new edition). Let ν be a signed measure on (X, \mathcal{M}) .

- (a) $L^1(\nu) = L^1(|\nu|)$.
- (b) If $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$.
- (c) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$.

Proof. Let ν be a signed measure on (X, \mathcal{M}) .

- (a) Let $f \in L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$. Then $0 \leq \int |f| d\nu^+ < \infty$ and $0 \leq \int |f| d\nu^- < \infty$, so

$$0 \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu| < \infty,$$

and $f \in L^1(|\nu|)$.

Let $f \in L^1(|\nu|)$, then $0 \leq \int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$, and the two integrals on the right are non-negative, so $0 \leq \int |f| d\nu^+ < \infty$ and $0 \leq \int |f| d\nu^- < \infty$, so $f \in L^1(\nu^+)$ and $f \in L^1(\nu^-)$. Therefore $f \in L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$.

This shows $L^1(\nu) = L^1(|\nu|)$.

(b) Let $f \in L^1(\nu)$, then

$$\left| \int f d\nu \right| = \left| \int f d\nu^+ - \int f d\nu^- \right| \leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right|,$$

and since ν^+, ν^- are positive measures and $|f| \geq f$,

$$\left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|,$$

so $|\int f d\nu| \leq \int |f| d|\nu|$.

(c) Let $E \in \mathcal{M}$. By (b),

$$\sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\} \leq \sup \left\{ \int_E |f| d|\nu| : |f| \leq 1 \right\}.$$

Since $|f| \leq 1$, $\int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E)$, so

$$\sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\} \leq |\nu|(E).$$

Let $X = P \dot{\cup} N$ be a Hahn decomposition for ν , then

$$\begin{aligned} |\nu|(E) &= \int_E d|\nu| = \int_E d\nu^+ + \int_E d\nu^- = \int_E d\nu^+ - \int_E (-1) d\nu^- \\ &= \int_{E \cap P} 1 d\nu^+ + \int_{E \cap N} 1 d\nu^+ - \left[\int_{E \cap P} (-1) d\nu^- + \int_{E \cap N} (-1) d\nu^- \right] \\ &= \int_{E \cap P} 1 d\nu^+ - \int_{E \cap N} (-1) d\nu^- = \int (\chi_{E \cap P} - \chi_{E \cap N}) d\nu. \end{aligned}$$

Since $|\chi_{E \cap P} - \chi_{E \cap N}| = 1$,

$$|\nu|(E) \leq \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\},$$

and the inequality in the opposite direction is known, so the two quantities are equal. \square

Exercise 4, pg 88 (in the new edition). If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Let (X, \mathcal{M}, ν) be a signed measure space, and λ, μ be positive measures such that $\nu = \lambda - \mu$. Let $X = P \dot{\cup} N$ be a Hahn decomposition for ν . Then for all $S \in \mathcal{M}$,

$$\nu(S \cap P) = \nu^+(S \cap P) = \lambda(S \cap P) - \mu(S \cap P) \Rightarrow \lambda(S \cap P) = \nu^+(S \cap P) + \mu(S \cap P) \geq \nu^+(S \cap P),$$

and likewise

$$\nu(S \cap N) = -\nu^-(S \cap N) = \lambda(S \cap N) - \mu(S \cap N) \Rightarrow \mu(S \cap N) = \nu^-(S \cap N) + \lambda(S \cap N) \geq \nu^-(S \cap N),$$

so

$$\nu^+(S) = \nu^+(S \cap P) \leq \lambda(S \cap P) \leq \lambda(S),$$

and

$$\nu^-(S) = \nu^-(S \cap N) \leq \mu(S \cap N) \leq \mu(S),$$

showing $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. \square