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Math 6338 HW 1

Exercise 3, pg. 118 (pg. 111). Every metric space is normal. [If A, B are closed sets in the metric space (X, ρ), consider the sets of points x where $\rho(x, A) < \rho(x, B)$ or $\rho(x, A) > \rho(x, B)$.]

Proof. Let (X, ρ) be a metric space, and A, B be closed in $(X, \mathcal{T}_\rho)$. Let $\mathcal{A} = \{x : \rho(x, A) < \rho(x, B)\}$ and $\mathcal{B} = \{x : \rho(x, B) < \rho(x, A)\}$. Let $x, y, z \in X$, then

$$\begin{aligned} d(y, z) \le d(y, x) + d(x, z) &\Rightarrow \inf_{z \in \mathcal{A}} d(y, z) \le d(y, x) + \inf_{z \in \mathcal{A}} d(x, z) \\ &\Rightarrow d(y, A) \le d(y, x) + d(x, A) \end{aligned}$$

likewise, $d(x, A) \le d(y, x) + d(y, A)$, so $|d(y, A) - d(x, A)| \le d(y, x)$.

Let $x \in \mathcal{A}$, $\epsilon = d(x, B) - d(x, A)$, and $y \in B(x, \frac{\epsilon}{4})$. Then

$$\begin{aligned} |d(y, A) - d(x, A)| \le \frac{\epsilon}{4} &\Rightarrow d(y, A) \le d(x, A) + \frac{\epsilon}{4} \\ |d(y, B) - d(x, B)| \le \frac{\epsilon}{4} &\Rightarrow d(y, B) \ge d(x, B) - \frac{\epsilon}{4} \end{aligned}$$

so

$$d(y, B) - d(y, A) \ge d(x, B) - \frac{\epsilon}{4} - d(x, A) - \frac{\epsilon}{4} = \frac{\epsilon}{2},$$

and $y \in \mathcal{A}$. Therefore $B(x, \frac{\epsilon}{4}) \subseteq \mathcal{A}$, so $\mathcal{A}$ is open. Likewise $\mathcal{B}$ is open.

Since $A \subset \mathcal{A}$, $B \subset \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$, X is normal.

□

Exercise 5, pg. 118 (pg. 112). Every separable metric space is second countable.

Proof. Let (X, ρ) be a separable metric space, and $E = \{e_n\} \subset X$ a countable dense subset of X. For each $e \in E$, let $N_e = \{B(e, \frac{1}{n})\}_{n \in \mathbb{N}}$; let $V = \cup_{e \in E} N_e$.

Let $x \in X$ and U be an open set such that $x \in U$. By the first countability of X and the fact $X = \overline{E}$, there is a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$. Let

$$U_n = B\left(x_n, \frac{1}{j}\right) \in N_{x_n},$$

where $j \in \mathbb{N}$ and $\rho(x_n, x) < \frac{1}{j} < 2\rho(x_n, x)$, so $U_n \in V$ and $x \in U_n$. Then let U be an open set such that $x \in U$; there is a $k \in \mathbb{N}$ such that $B(x, \frac{1}{k}) \subset U$. Take $n \in \mathbb{N}$ such that

$$\rho(x_n, x) < \frac{1}{3k},$$

then for all $y \in U_n$,

$$\rho(y, x) \le \rho(x_n, x) + \rho(y, x_n) < \frac{1}{3k} + \frac{2}{3k} = \frac{1}{k},$$

so $U_n \subset B(x, \frac{1}{k}) \subseteq U$. Since $x \in U_n$ for all $n \in \mathbb{N}$ and for each U open such that $x \in U$ there is a $U_n \subset U$, $\{U_n\}_{n \in \mathbb{N}}$ is a neighborhood base of $x \in X$.

Since V is countable and contains a neighborhood base of x for each $x \in X$, X is second countable. □

Exercise 16, pg. 123 (16, pg. 117). Let X be a topological space, Y a Hausdorff space, and f, g continuous maps from X to Y.

(a) $\{x : f(x) = g(x)\}$ is closed.

(b) If $f = g$ on a dense subset of X, then $f = g$ on all of X.

Exercise 18, pg. 123 (19, pg. 117). If X and Y are topological spaces and $y_0 \in Y$, then X is homeomorphic to $X \times \{y_0\}$ where the latter has the relative topology as a subset of $X \times Y$.