Alex Gittens
February 22, 2006

Math 6338 HW 3

Exercise 8, pg. 156 (exer. 6, pg. 146). Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ which makes $M(X)$ into a Banach space. [Use Theorem (5.1).]

Proof. Let $(X, \mathcal{M})$, $M(X)$, and $\|\cdot\|$ be as stated. Since for all complex measures ν, $|\nu|$ is a finite measure, $\|\cdot\| : M(X) \to [0, \infty)$.

Let $\nu = \nu_r + i\nu_i \in M(X)$, and assume $|\nu|(X) = 0$, then

$$\nu \ll |\nu| \Rightarrow \nu_i, \nu_r \ll |\nu| \Rightarrow |\nu_i|, |\nu_r| \ll |v| \Rightarrow |\nu_i|(X) = |\nu_r|(X) = 0,$$

so by Exercise 2, §3.2 X is ν_i-null and ν_r-null, so X is ν-null. That is, for all $E \subset X$, $E \in \mathcal{M}$, $\nu(E) = 0$; so $\nu = 0$. If $\nu = 0$, then clearly $|\nu|(X) = 0$. Therefore, $\|\nu\| = 0 \Leftrightarrow \nu = 0$.

Let $\nu_1, \nu_2 \in M(X)$, then by Proposition 3.14, $|\nu_1 + \nu_2| \leq |\nu_1| + \nu_2|$, so $\|\nu_1 + \nu_2\| \leq \|\nu_1\| + \|\nu_2\|$.

Let $\lambda \in \mathbb{C}$, $\nu \in M(X)$, and μ be a σ-finite positive measure. Since $|\nu|$ is defined to be the measure satisfying $d|v| = |f|\, d\mu$ when $d\nu = f\, d\mu$,

$$\lambda\nu = \int \lambda f\, d\mu \Rightarrow d(\lambda\nu) = (\lambda f)\, d\mu \Rightarrow d|\lambda\nu| = |\lambda f|\, d\mu$$

$$|\lambda\nu|(X) = \int_X |\lambda f|\, d\mu = |\lambda| \int_X |f|\, d\mu \Rightarrow \|\lambda\nu\| = |\lambda|\|\nu\|.$$

Therefore $\|\cdot\|$ is a norm on $M(X)$.

Let $\{\nu_n\} \subset M(X)$ be such that $\sum_{n=1}^{\infty} \|\nu_n\| < \infty$. Then for all $E \in \mathcal{M}$,

$$\left|\sum_{n=1}^{\infty} \nu_n(E)\right| \leq \sum_{n=1}^{\infty} |\nu_n(E)| \leq \sum_{n=1}^{\infty} |\nu_n|(E) \leq \sum_{n=1}^{\infty} |\nu_n|(X) = \sum_{n=1}^{\infty} \|\nu_n\| < \infty,$$

so $\nu = \sum_{n=1}^{\infty} \nu_n : \mathcal{M} \to \mathbb{C}$.

If further, $\{E_j\} \subset \mathcal{M}$ are disjoint, then

$$\nu(\cup_{j=1}^{\infty} E_j) = \sum_{n=1}^{\infty} \nu_n(\cup_{j=1}^{\infty} E_j) = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} \nu_n(E_j) \right),$$

and since the E_j are disjoint,

$$|\nu(\cup_{j=1}^{\infty} E_j)| \leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |\nu_n|(E_j) \leq \sum_{n=1}^{\infty} \|\nu_n\| < \infty,$$

so Fubini can be applied, giving

$$\nu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \nu_n(E_j) = \sum_{j=1}^{\infty} \nu(E_j).$$

Also, $\nu(\emptyset) = \sum_{j=1}^{\infty} \nu_n(\emptyset) = 0$, so $\nu \in M(X)$.

Finally,

$$\|\nu - \sum_{n=1}^{N} \nu_n\| = \| \sum_{n=N+1}^{\infty} \nu_n\| \leq \sum_{n=N+1}^{\infty} \|\nu_n\| \to 0$$

since $\sum_{n=1}^{\infty} \|\nu_n\| < \infty$; therefore $\sum_{n=1}^{N} \nu_n \to \nu$ as $N \to N$. That is, absolutely convergent series in $M(X)$ converge.

By Theorem 5.1, $M(X)$ is complete with respect to the norm $\|\nu\| = |\nu|(X)$, and it is a vector space, so it is a Banach space. □

Exercise 12(b,c) pg. 156 (exer. 11, pg. 147). Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper closed subspace.

(a) $\|x + \mathcal{M}\| = \inf\{\|x + y\| : y \in \mathcal{M}\}$ is a norm on $\mathcal{X}/\mathcal{M}$, called the quotient norm.
(b) For any $\epsilon > 0$ there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + \mathcal{M}\| \geq 1 - \epsilon$.
(c) The projection $\pi(x) = x + \mathcal{M}$ from $\mathcal{X}$ to $\mathcal{X}/\mathcal{M}$ has norm 1.
(d) If $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$. [Use Theorem (5.1).]
(e) The topology defined by the quotient norm is the quotient topology as defined earlier in Exercise 28, §4.2.

Proof. b) The statement is trivially true for $\epsilon \geq 1$, because for all $x \in X$, $\|x + \mathcal{M}\| \geq 0 \geq 1 - \epsilon$, so consider $1 > \epsilon > 0$. The fact $\mathcal{M}$ is a closed proper subspace of X implies $\mathcal{M}$ is not dense in X, so there is an $x_0 \in X$ such that $\|x_0 + \mathcal{M}\| > 0$. Therefore, there is a sequence $\{y_n\} \subset \mathcal{M}$ such that $\|x_0 - y_n\| \searrow \|x_0 + \mathcal{M}\|$.

For all $\delta > 0$, there is an N_δ such that $n \geq N_\delta \Rightarrow 0 < \|x_0 - y_n\| - \|x_0 + \mathcal{M}\| < \delta$. Let $v = \frac{x_0 - y_{N_\delta}}{\|x_0 - y_{N_\delta}\|}$, then $\|v\| = 1$, and

$$\|v + \mathcal{M}\| = \left\| \frac{x_0 - y_{N_\delta}}{\|x_0 - y_{N_\delta}\|} + \mathcal{M} \right\| = \frac{1}{\|x_0 - y_{N_\delta}\|} \|x_0 - y_{N_\delta} + \mathcal{M}\| = \frac{1}{\|x_0 - y_{N_\delta}\|} \|x_0 + \mathcal{M}\|.$$

Since $\|x_0 - y_{N_\delta}\| < \|x_0 + \mathcal{M}\| + \delta$,

$$\|v + \mathcal{M}\| > \frac{\|x_0 + \mathcal{M}\|}{\|x_0 + \mathcal{M}\| + \delta} > 1 - \epsilon$$

for an appropriately small choice of δ.

Therefore, for any $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + \mathcal{M}| \geq 1 - \epsilon$.

c) Clearly, (b) above implies $\|\pi\| \geq 1$. Also, $0 \in \mathcal{M}$, so if $x \in X$ and $\|x\| = 1$,

$$\|\pi(x)\| = \|x + \mathcal{M}\| = \inf\{\|x + y\| : y \in \mathcal{M}\} \leq \|x + 0\| = \|x\| = 1,$$

so $\|\pi\| \leq 1$. Therefore $\|\pi\| = 1$. □

Exercise 13, pg. 156 (exer. 12, pg. 148). If $\|\cdot\|$ is a seminorm on $\mathcal{X}$, let $\mathcal{M} = \{x \in X : \|x\| = 0\}$. Then $\mathcal{M}$ is a subspace, and $\|x + \mathcal{M}\| = \|x\|$ defines a norm on $\mathcal{X}/\mathcal{M}$.

Proof. Let X, $\|\cdot\|$, $\mathcal{M}$ be as stated, with $\mathbb{F}$ the field associated with X. Then $0 \in \mathcal{M}$, so $\mathcal{M} \neq \emptyset$. If $x, y \in \mathcal{M}$ and $\lambda \in \mathbb{F}$, then

$$\|x + \lambda y\| \leq \|x\| + |\lambda|\|y\| = 0,$$

so $x + \lambda y \in \mathcal{M}$, and $\mathcal{M}$ is a subspace of X.

If $x + \mathcal{M} = y + \mathcal{M}$, then $x - y = m \in \mathcal{M}$. Therefore

$$\|x\| = \|y + m\| \leq \|y\| + \|m\| = \|y\|$$
$$\|y\| = \|x - m\| \leq \|x\| + \|m\| = \|x\|,$$

so $\|y\| = \|x\| \Rightarrow \|x + \mathcal{M}\|_{X/\mathcal{M}} = \|y + \mathcal{M}\|_{X/\mathcal{M}}$, so $\|\cdot\|_{X/\mathcal{M}} : X/\mathcal{M} \to [0, \infty)$ is well-defined.

Also,

$$\|(x + y) + \mathcal{M}\|_{X/\mathcal{M}} = \|x + y\| \leq \|x\| + \|y\| = \|x + \mathcal{M}\|_{X/\mathcal{M}} + \|y + \mathcal{M}\|_{X/\mathcal{M}},$$

so the triangle inequality holds.

If $\lambda \in \mathbb{F}$,

$$\|\lambda x + \mathcal{M}\|_{X/\mathcal{M}} = \|\lambda x\| = |\lambda|\|x\| = |\lambda|\|x + \mathcal{M}\|_{X/\mathcal{M}},$$

and

$$\|x + \mathcal{M}\|_{X/\mathcal{M}} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x \in \mathcal{M} \Leftrightarrow x + \mathcal{M} = 0_{X/\mathcal{M}}.$$

Therefore $\| \cdot \|_{X/\mathcal{M}}$ is a norm on $X/\mathcal{M}$. □

Exercise 15, pg. 156 (exer. 14, pg. 148). Suppose $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces and let $T \in L(\mathcal{X}, \mathcal{Y})$.

(a) $\mathcal{N}(T) = \{x : Tx = 0\}$ is a closed subspace of $\mathcal{X}$.

(b) If $\mathcal{M}$ is a closed subspace of $\mathcal{N}(T)$, there is a unique $S \in L(\mathcal{X}/\mathcal{M}, \mathcal{Y})$ such that $T = S \circ \pi$ where $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is the projection. Moreover, $\|S\| = \|T\|$.

Proof. (a) Let X, Y, T, $\mathcal{N}(T)$ be as given, and let $\{x_n\} \subseteq \mathcal{N}(T)$, with $x_n \to x \in X$. Then

$$\|Tx\| = \|Tx - Tx_n\| = \|T(x - x_n)\| \leq \|T\|\|x - x_n\| \to 0,$$

so $\|Tx\| = 0 \Rightarrow Tx = 0$, and $x \in \mathcal{N}(T)$. Also, $0 \in \mathcal{N}(T)$, so $\mathcal{N}(T) \neq \emptyset$, and if $x, y \in \mathcal{N}(T)$ and λ is a scalar, then

$$T(x + \lambda y) = Tx + \lambda Ty = 0 + \lambda 0 = 0,$$

so $x + \lambda y \in \mathcal{N}(T)$. Therefore $\mathcal{N}(T)$ is a closed subspace of X.

(b) Let $\mathcal{M}$ be a closed subspace of $\mathcal{N}(T)$, and define $S : X/\mathcal{M} \to Y$ by $S(x + \mathcal{M}) = Tx$. Then S is well-defined: if $x + \mathcal{M} = y + \mathcal{M}$, then $x - y = m \in \mathcal{M}$, so

$$S(x + \mathcal{M}) = Tx = T(y + m) = Ty + Tm = Ty = S(y + \mathcal{M}),$$

and $T = S \circ \pi$: for all $x \in \mathcal{M}$,

$$(S \circ \pi)(x) = S(x + \mathcal{M}) = Tx.$$

Clearly S is linear: if $x, y \in \mathcal{M}$ and λ is a scalar,

$$S(x + \lambda y + \mathcal{M}) = T(x + \lambda y) = Tx + \lambda Ty = S(x + \mathcal{M}) + \lambda S(y + \mathcal{M}).$$

Also, since $\|x + \mathcal{M}\| = 1$ implies there is a sequence $\{y_n\} \in \mathcal{M}$ such that $\|x - y_n\| \searrow \|x + \mathcal{M}\| = 1$, if $\|x + \mathcal{M}\| = 1$, then

$$\|Tx\| = \|Tx - Ty_n\| = \|T(x - y_n)\| \leq \|T\|\|x - y_n\| \to \|T\|,$$

so

$$\|S\| = \sup\{\|S(x + \mathcal{M})\| : \|x + \mathcal{M}\| = 1\} = \sup\{\|Tx\| : \|x + \mathcal{M}\| = 1\} \leq \|T\|,$$

and since $S \in L(X/\mathcal{M}, Y)$ and $\pi \in L(X, X/M)$,

$$\|T\| = \|S \circ \pi\| \leq \|S\|\|\pi\| \leq \|S\|,$$

so $\|T\| = \|S\|$.

Therefore $S \in L(X/\mathcal{M}, Y)$ is the unique function satisfying $T = S \circ \pi$.

□