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Math 6338 HW 5

Exercise 63a, pg. 178 (pg. 170). Let $\mathcal{X}$ be an infinite-dimensional Hilbert space. Then every orthonormal sequence in $\mathcal{X}$ converges weakly to zero.

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ be an orthonormal sequence, and let $f \in \mathcal{X}^\star$. Then there is a $y \in \mathcal{X}$ such that $f(x) = \langle x, y\rangle$ for $x \in \mathcal{X}$. Applying Bessel's inequality,

$$\sum_{n\in\mathbb{N}} |\langle y, x_n\rangle|^2 \leq \|y\|^2,$$

gives $f(x_n) = \overline{\langle y, x_n\rangle} \to 0 = f(0)$ as $n \to \infty$. Since this is true for all $f \in \mathcal{X}^\star$, $x_n \to 0$ weakly. □

Exercise 9, pg. 187 (pg. 179). If $\|f_n - f\|_p \to 0$ where $p < \infty$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n where $p < \infty$, then $\|f_n - f\|_p \to 0$.

Proof. Let $\|f_n - f\|_p \to 0$ with $p < \infty$. Given $\epsilon > 0$, let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| > \epsilon\}$, then

$$\|f_n - f\|_p^p = \int |f_n - f|^p \geq \int_{E_{n,\epsilon}} |f_n - f|^p > \epsilon^p \mu(E_{n,\epsilon}) \Rightarrow \epsilon^{-p}\|f_n - f\|_p^p > \mu(E_{n,\epsilon}),$$

so $\|f_n - f\|_p \to 0$ implies $f_n \to f$ in measure.

Let $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n, with $p < \infty$. Then by Theorem 2.30, there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. Therefore, $|f| \leq g$ a.e., so $f \in L^p$, so $f_n - f \in L^p$ for all n. Since $f_{n_k} - f \to 0$ a.e. and $|f_{n_k} - f|^p \leq 2^p|g|^p$ a.e., by the DCT,

$$\|f_{n_k} - f\|_p^p = \int |f_{n_k} - f|^p \to \int 0 = 0.$$

By the above procedure, given a subsequence of $\{f_n\}$, one can find a subsequence $\{f_{n'}\}$ of the given one which satisfies

$$\|f_{n'} - f\|_p \to 0.$$

Assume this does not hold for the original sequence $\{f_n\}$, i.e. $\|f_n - f\|_p \not\to 0$, then there is a $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that $\|f_{n_k} - f\|_p > \epsilon$ for all n_k. But then no subsequence of $\{f_{n_k}\}$ converges to f in L^p. This contradiction implies that the original sequence satisfies

$$\|f_n - f\|_p \to 0.$$

□

Exercise 22a, pg. 192 (pg. 184). Let $X = [0, 1]$, with Lebesgue measure. Let $f_n(x) = \cos 2\pi nx$. Then $f_n \to 0$ weakly in L^2 (cf. Exercise 5.63), but $f_n \not\to 0$ a.e. or in measure.

Proof. Let $m, n \in \mathbb{N}$, then

$$\begin{aligned}\langle \sqrt{2}f_n, \sqrt{2}f_m \rangle = 2\int_{[0,1]} \cos(2\pi nx)\cos(2\pi mx)\,dx = \\ = \int_{[0,1]} \cos(2\pi(n-m)x) + \cos(2\pi(n+m)x)\,dx \\ = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases},\end{aligned}$$

that is, $\{\sqrt{2}f_n\}$ is an orthonormal sequence in $L^2(X)$, so $\sqrt{2}f_n \to 0$ weakly by the first exercise, or $\forall y \in L^2(X)$, $\langle \sqrt{2}f_n, y\rangle = \sqrt{2}\langle f_n, y\rangle \to 0$, so $f_n \to 0$ weakly also.

Let $E_{n,\frac{1}{\sqrt{2}}} = \left\{x : |f_n(x)| > \frac{1}{\sqrt{2}}\right\}$. Note that f_n consists of n periods of the $\frac{1}{n}$ periodic restriction of f_n to $[0, \frac{1}{n}]$, so

$$\mu\left(E_{n,\frac{1}{\sqrt{2}}}\right) = n\mu\left(\left\{x : x \in \left[0, \frac{1}{n}\right], |f_n(x)| > \frac{1}{\sqrt{2}}\right\}\right),$$

and

$$\left\{x : x \in \left[0, \frac{1}{n}\right], |\cos(2\pi nx)| > \frac{1}{\sqrt{2}}\right\} = \left[0, \frac{1}{8n}\right] \cup \left[\frac{3}{8n}, \frac{5}{8n}\right] \cup \left[\frac{7}{8n}, \frac{1}{n}\right],$$

so $\mu(E_{n,\frac{1}{\sqrt{2}}}) = n \times 2 \times \frac{2}{8n} = \frac{1}{2}$. Since $\mu(E_{n,\frac{1}{\sqrt{2}}}) \not\to 0$ as $n \to \infty$, $f_n \not\to 0$ in measure.

Assume $f_n \to 0$ a.e., then $|f_n| \leq 1$ implies by the DCT that

$$\int_X |f_n| \to \int 0 = 0,$$

but

$$\int_X |f_n| = \sum_{c=1}^{n} \int_{[0,\frac{1}{n}]} |\cos 2\pi nx| = \sum_{c=1}^{n} \frac{2}{n\pi} = \frac{2}{\pi}.$$

This contradiction shows $f_n \not\to 0$ a.e.

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