NOTES FROM INTRO TO COMPLEX ANALYSIS (MATH 3364) SPRING 2005

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If each of x and y is a number, we associate a number denoted by $x+y$ with the following properties:

A1. $x+y$ is $y+x$
A2. if each of x,y,and z is a number, $x+(y+z)$ is $(x+y)+z$
A3. there is a number 0 such that $x+0$ is x
A4. if x is a number, then there is a number y such that $x+y$ is 0

Prob. 1. If N is a number such that $x+N$ is x, then N is 0.

Prob. 2. If x is a number and w is a number such that $x+w$ is 0, then w is y and we call this number the negative of x and write it $-x$.

If each of x and y is a number, then we associate a number denoted by xy with the following properties:

M1. xy is yx
M2. if each of x,y, and z is a number, then $x(yz)$ is $(xy)z$
M3. there is a number w such that xw is x for every number x
M4. if x is a number distinct from 0, then there is a number y such that xy is w

Additional properties connecting multiplication and addition:

C1. 0 is not 1
C2. if each of x, y, and z is a number, $x(y+z)$ is $xy+xz$

Prob. 3. If z is a number such that xz is x for every number x, then z is w and we denote it by 1, read one.

Prob. 4. If x is a number distinct from 0 and z is a number such that xz is 1, then z is y and we denote it by $\frac{1}{x}$, read the reciprocal of x.

Def. 1. The statement that P is a point means P is an ordered pair, written (a,b). a is the first number and b is the second number.

Notation. The set of all numbers is called the number line, the set of all points the plane.
Prob. 5. Given the addition of points defined by $(a,b)+(c,d)$ is $(a+c,b+d)$, show that this operation satisfies A1-A4.

Prob. 6. Given the multiplication of points defined by $(a,b)(c,d)$ is $(ac-bd,ad+bc)$, show that this operation satisfies M1-M4, and along with addition of points, satisfies C1 and C2.

Def. 2. The plane with $+$ and "$\cdot$" is called the complex plane.

Def. 3. If $P=(a,b)$ and $Q=(c,d)$ are points, then the distance from P to Q is the number $(a-c)^2+(b-d)^2$ and is denoted by $|P-Q|$.

Def. 4. The statement that R is a region means there is a positive number r and a point P, and R is the set to which x belongs only in case x is a point and $|x-P|<r$. We denote R by $C_r(P)$.

Prob. 7. If $C_r(P)$ has a point in common with $C_p(Q)$, then there is a region that is a subset of both $C_r(P)$ and $C_p(Q)$.

Prob. 8. Sketch the set K where K is $\left\{(x,x)(x,x) : \frac{1}{2} \leq x \leq 2\right\}$.

Prob. 9. Sketch the set M where M is $\{(x,y)(x,y) : x \geq 0\}$.

Def. 5. Suppose M is a point collection. The statement that P is a limit point of M means P is a point and if R is a region containing P, then R contains a point of M distinct from P.

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Def. 6. If z is (a,b), then $z^\star$ is $(a,-b)$.

Def. 7. If P is the point (a,b) and c is a number, then $cP = (ca, cb)$.

Notation. Denote $(0,0)$ by 0, $(1,0)$ by 1, and $(0,1)$ by i. Then if $z = (a,b)$, z is $a + bi$.
Def. 8. The statement that f is a function means f is an ordered number pair collection, no two of which have the same first number. The initial set of f is the set to which x belongs only in case x is the first number of an ordered pair in f and the final set is the set to which y belongs only in case y is the second number of an ordered pair in f.

Def. 9. The statement that f is a point function from the numbers means f is an ordered pair collection, the first term of each pair in f is a number, the second term is a point, and no two pairs have the same first number.

Def. 10. The statement that f is a point function means f is an ordered point pair collection, no two pairs having the same first point.

Def. 11. The statement that S is a number sequence means S is a function with initial set the nonnegative integers.

Def. 12. The statement that T is a point sequence means T is a point function from the numbers with initial set the nonnegative integers.

Notation. If S is a number sequence, we often write $S_1, S_2, \ldots$ instead of $S(1), S(2), \ldots$, i.e. S_t is $S(t)$ for each nonnegative integer t.
Def. 13. Suppose $a_1, a_2, \ldots$ is a number sequence. The statement that $a_1, a_2, \ldots$ converges means there is a number A and if c is a positive number, then there is a positive integer N such that if n is an integer greater than N, then $|a_n - A| < c$. We say that $a_1, a_2, \ldots$ converges to A.

Prob. 10. Suppose $a_1, a_2, \ldots$ is a number sequence and $b_1, b_2, \ldots$ is the number sequence defined by $b_0 = a_0$, $b_1 = a_0 + a_1$, $b_2 = a_0 + a_1 + a_2, \ldots$, $b_n = a_0 + \ldots + a_n$. If the sequence $b_0, b_1, \ldots$ converges, then the sequence $a_0, \ldots$ converges to zero.

Prob. 11. If z is a point and $|z| < 1$, then the point sequence $1, 1+z, 1+z+z^2, 1+z+z^2+z^3+\ldots$ converges.

Def. 14. The statement that a point collection M is bounded means there is a region R such that M is a subset of R.

Notation. $|P - Q| = \left[(p_1 - q_1)^2 + (p_2 - q_2)^2\right]^{\frac{1}{2}}$ and $P - (0,0) = P$, so $|P - (0,0)| = |P|$.
Prob. 12. Suppose S is a point sequence with bounded final set. If z is a point such that $|z| < 1$, then the sequence $s_0, s_0 + s_1 z, s_0 + s_1 z + s_2 z^2, \ldots$ converges.

Jan 25

Def. 15. Suppose $[a, b]$ is a number interval. The statement that S is a subdivision of $[a, b]$ means S is a finite non-decreasing number sequence having an odd number of terms, the first of which is a and the last is b.

Def. 0. Suppose M is a set. The statement that R is a relation on M means R is an ordered element pair collection, with each element from M; moreover, the statement that S is a contraction of R means S is a subset of R.

Def. 16. Suppose S is a subdivision of $[a, b]$. The statement that R is a refinement of S means it is a subdivision of $[a, b]$ and each term in S with an odd subscript is a term in R with an odd subscript.

Prob. 13. If each of S and T is a subdivision of $[a, b]$, then there is a common refinement R.

Feb 7

Prob. 14. Can you find a point z such that $z \neq (1, 0)$ and $z^3 = 1$? Moreover, if $n > 3$ is a positive integer, is there a point z, $z \neq (1, 0)$, such that $z^n = 1$?

Prob. 15. Suppose L is a line containing $(0, 0)$ and for each point $f(z) = z^2$ such that f is a function from the plane into the plane. Sketch $f(L)$.

Feb 9

Def. 17. Suppose f is a point function. The statement that f is continuous at (P, Q) means (P, Q) is in f and if S is a region containing Q, there is a region R containing P such that if z is a point in the initial set of f that is in R, then $f(z)$ is in S.

Def. 18. Suppose each of f and g is a point function. The statement that h is $f + g$ means the initial set of f is the initial set of g and h is the point function to which (u, v) belongs only in case u is in the initial set of f and $v = f(u) + g(u)$.

Prob. 16. If each of f and g is a point function, $h = f + g$ and f is continuous at (u, v) and g is continuous at (u, w), then h is continuous at $(u, v + w)$.

Def. 19. Suppose f is a point function. The statement that f is continuous means if (u, v) belongs to f then f is continuous at (u, v).

Feb 15

Prob. 17. If f is an increasing function from the interval $[0, 1]$ onto $[0, 1]$, then f is continuous, moreover f^{-1}, the inverse of f, is continuous.

Def. 20. Suppose f is a point function with initial set containing the number interval $[a, b]$. The statement that f is a B.V. function on $[a, b]$ means there is a number W such that if $a_1, a_2, \ldots, a_{n+1}$ is a finite nondecreasing number sequence the first term of which is a and the last term of which is b, then

$$\sum_{t=1}^{n} |f(a_{t+1}) - f(a_t)| \leq W.$$

The least such number W is denoted by $\bigvee_a^b f$.

Def. 21. The statement that P is a path means

(i) There is a function g with initial set the interval $[a, b]$ and final set a subset of the plane such that g is B.V. on $[a, b]$ and continuous on $[a, b]$.

(ii) P is a function collection to which k belongs only in case there is an increasing function h from $[a, b]$ onto $[a, b]$ such that $k = g(h)$. We call the final set of g the carrier of P.

The length of the path P is defined to be

$$l(P) = \bigvee_0^1 g,$$

where g is in P.

Def. 22. Function inverses? T.B.A.

Feb 21

Def. 23. Suppose each of f and g is a function from the numbers into the plane and $[a, b]$ is a number interval that is a subset of the initial set of each of f and g. The statement that f is g-integrable on $[a, b]$ means there is a point M such that if c is a positive number, then there is a subdivision $S = \{s_i\}_{i=1}^{2n+1}$ of $[a, b]$ such that if $R = \{r_i\}_{i=1}^{2m+1}$ is a refinement of S then

$$\left| M - \sum_{t=1}^{m} f(r_{2t}) \left[g(r_{2t+1}) - g(r_{2t-1}) \right] \right| < c.$$

We denote M by the symbol $\int_a^b f\, dg$.

Prob. 18. Suppose each of f, g and h is a function and c is a number. If the interval $[a, b]$ is a subset of the initial set of each of f, g and h and

$$\int_a^b f\, dh, \quad \int_a^b g\, dh \qquad \text{exist, then}$$

$\int_a^b cf + g\, dh = c\int_a^b f\, dh + \int_a^b g\, dh$ exists, moreover

$$\int_a^b f\, dg = f(b)g(b) - f(a)g(a) - \int_a^b g\, df.$$

Prob. 19. If f is a continuous function from the numbers into the plane and g is a continuous function from the plane into the plane such that the initial set of g contains the final set of f, then $\hat{g}(f)$ is a continuous function from numbers into the plane, where $\hat{g}$ is the contraction of g to the final set of f.

Prob. 20. Suppose each of f and g is a function from numbers into the plane such that the interval $[a, b]$ is a subset of the initial set of each of f and g. If f is continuous on $[a, b]$ and g is a B.V. function on $[a, b]$, then $\int_a^b f\, dg$ exists.

Feb 24

Prob. 21. If $x_1, x_2, \ldots$ is a point sequence such that if $c > 0$ then there is a positive number N such that if m and n are integers greater than N, then $|x_m - x_n| < c$, then the sequence $x_1, x_2, \ldots$ converges.

Def. 24. Suppose each of H and K is a set of points. The statement that H and K are mutually separated means neither contains a point or a limit point of the other.

Prob. 22. If $f(x) = \frac{1}{x}$ for $x \geq 1$ and H is the x-axis, are f and H mutually separated? If $f + H$ connected?

Prob. 23. If $g(x) = \sin\left(\frac{1}{x}\right)$ for $0 < x \leq 1$, $V = \{(0, u) : -1 \leq u \leq 1\}$, are g and V mutually separated? If $g + V$ connected?

Def. 25. The statement that a set M is connected means M is not the sum of two mutually separated sets.

Def. 26. The statement that a set M is open means no point of M is a limit point of points not in M.

Prob. aside1. Is a region connected? Is a region open?

Prob. aside2. Notice that for a sphere the volume is $V = \frac{4}{3}\pi r^3$ and the surface area is $A = 4\pi r^2$, and for a circle, the area $A = \pi r^2$ and the circumference $C = 2\pi r$. What other figures and/or bodies satisfy the relationship $V' = A$, or $A' = C$ for appropriate parameterizations?

Mar 3

Def. 27. Suppose M is a number collection. The statement that P is a Q-point of M means P is a number and every interval containing P contains a point of M distinct from P.

Prob. 24. If each of H and K is a number collection and P is a Q-point of $H \cup K$, then P is a Q-point of H or P is a Q-point of K.

Prob. 25. Suppose $a_0, a_1, \ldots$ is a bounded sequence. Mr. Gittens showed that if z is a point, $|z| < 1$, then the sequence $s_0(z), s_1(z), \ldots$ converges, where

$$s_0(z) = a_0, s_1(z) = a_0 + a_1 z, s_2(z) = a_0 + a_1 z + a_2 z^2, \ldots.$$

Suppose now that f is a function with initial set that contains all points z, $|z| < 1$, and for each z, $|z| < 1$, $s_0(z), s_1(z), \ldots$ converges to $f(z)$. Show that if $0 < r < 1$ and $c > 0$, there is a positive integer N such that if $n > N$ and $|z| \leq r$,

$$|f(z) - s_n(z)| < c.$$

Mar 10

Prob. 26. If S is a connected point set having two points, then every point of S is a limit point of S.

Prob. 27. If f is a continuous point function and the initial set of f is connected, then the final set of f is connected.

Def. 28. Suppose A is a point set. A component of A is a connected subset of A that is not a proper subset of any connected subset of A.

Def. 29. If each of P and Q is a path and each function in each of P and Q has initial set the interval $[0, 1]$, then the statement that R is $P \oplus Q$ means if p is in P and q is in Q, then

i. $p(1)$ is $q(0)$
ii. R is the path of which r is a member, where $r(t) = p(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $r(t) = q(2t-1)$ for $\frac{1}{2} \leq t \leq 1$.

Def. 30. If each of A and B is a point, the statement that $[A; B]$ is the linear path from A to B means $[A; B]$ is the path of which one member is the contraction to the interval $[0, 1]$ of $(1 - j)A + jB$, where $j(x) = x$ for all numbers x.

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Def. 31. The statement that P is a polygonal path means there is a point sequence $P_0, P_1, P_2, \ldots, P_{n+1}$ such that $P = [P_0; P_1] \oplus [P_1; P_2] \oplus \cdots \oplus [P_n; P_{n+1}]$.

Def. 32. If K is a path such that each h in K has initial set the interval $[0, 1]$ and f is a point function with initial set containing K' (the carrier of K) then $\int_K f$ is the point $\int_0^1 f(z)\, dz$ where z is in K. This is called the Cauchy integral of f over the path K.

Def. 33. Suppose f is a point function. The statement that f has slope at the ordered pair (p, q) means (p, q) belongs to f, p is a limit point of the initial set of f and there is a point w such that if $c > 0$, then there is a $d > 0$ such that if $0 < |p - t| < d$ and t is in the initial set of f, then

$$\left| w - \frac{q - f(t)}{p - t} \right| < c.$$

w is called the slope of f at (p, q).

Prob. 28. If the point function f has slope at (p, q) then f is continuous at (p, q).

Def. 34. Suppose f is a point function with initial set A and B is the subset of A to which p belongs only in case f has slope at $(p, f(p))$, then the derivative of f, denoted by f' is the point function to which (u, v) belongs only in case u is in B and v is the slope of f at $(u, f(u))$.

Prob. 29. If f is a point function and K is a path from A to B in the initial set of f and f' is continuous on K', then

$$\int_K f' = f(B) - f(A).$$

EVERYTHING WITHOUT A DATE

Def. 35. The statement that f is analytic means f is a point function with initial set an open connected set Q and f has slope at $(z, f(z))$ for each z in Q. An entire function is an analytic function with initial set the plane.

Prob. 30. If f is analytic and $f'(z) = 0$ for all z in the initial set of f, then f is a constant function.

Prob. 31. Suppose z is a point $c > 0$, then

$$\int_{S_c(z)} \frac{1}{x - z}\, dx = 2\pi i,$$

where

$$S_c(z) = [z + c - ic; z + c + ic] \oplus [z + c + ic; z - c + ic] \oplus [z - c + ic; z - c - ic] \oplus [z - c - ic; z + c - ic]$$

and x is a function in $S_c(z)$.

Prob. 32. If Q is an open connected set and f is a continuous point function with initial set Q, then the following two statements are equivalent:

(1) There is an analytic function g such that $g' = f$

(2) If K is a closed path in Q, then

$$\int_K f = 0$$

Prob. 33.

$$\int_{S_c(z)} \frac{1}{I - z}\, dI = 2\pi i,$$

where I is the identity function on the plane.

Lemma a. If h is the contraction to $[-1, 1]$ of $\frac{j+i}{1+ij}$ (where j is the identity function on the number line, then h is reversible with final set W such that w is in W only in case $|w| = 1$ and $\text{Imag}(w) \geq 0$, and in this case, $h^{-1}(w) = \frac{w-i}{1-iw}$. π is defined as

$$\pi = 2 \int_{-1}^{1} \frac{1}{1 + j^2} dj.$$

Lemma b. If K is any of the four paths indicated in the definition of $S_c(z)$, then

$$\int_Z \frac{1}{I-z} = \frac{i\pi}{2}.$$

Prob. 34. There is an analytic function that is not the derivative of an analytic function.

Def. 36. If z is a point distinct from $(0,0)$, the principal argument of z, defined by $\operatorname{Arg}(z)$, is defined as

(1) if $\operatorname{Imag}(z) = 0$ and $\operatorname{Real}(z) > 0$, then $\operatorname{Arg}(z) = 0$;
(2) if $\operatorname{Imag}(z) = 0$ and $\operatorname{Real}(z) < 0$, then $\operatorname{Arg}(z) = \pi$;
(3) if $\operatorname{Imag}(z) > 0$ and $t = \frac{\operatorname{Real}(z)}{|z|+\operatorname{Imag}(z)}$, then

$$\operatorname{Arg}(z) = 2\int_t^1 \frac{1}{1+j^2}\,dj;$$

(4) if $\operatorname{Imag}(z) < 0$, then $\operatorname{Arg}(z) = -\operatorname{Arg}(z^\star)$.

Prob. 35. If x is a number and $-\pi < x \leq \pi$, then there is only one point w such that $|w| = 1$ and $\operatorname{Arg}(w) = x$.

Def. 37. If M is a point set and $c > 0$, then $I_c(M)$ denotes the point set to which p belongs only in case there is a point m in M such that $|p - m| \leq c$.

Prob. 36. If M is a closed and bounded subset of an open connected set Q, then there is a positive number c such that $I_c(M)$ is a subset of Q.

Prob. 37. Suppose f is a point function that is continuous on the open connected set Q and K is a path in Q. If $c > 0$ then there is a polygonal path P in Q such that

$$\left|\int_K f - \int_P f\right| < c$$

and $P = \bigoplus_{t=1}^n [y(u_{t+1}); y(u(t))]$, where y is in K, $u_1 = 0$, $u_{n+1} = 1$, and $u_t < u_{t+1}$ for $t = 1, 2, \ldots, n$.

Prob. 38. Suppose A, B, and C are three non-collinear points and W is the point set to which w belongs only in case there is a point p in $[B;C]'$ such that w is in $[A;p]'$. If f is a point function that has a derivative on W and $K = [A;B] \oplus [B;C] \oplus [C;A]$ (K is called a triangular paht), then

$$\int_K f = 0.$$

Lemma a. W is closed, bounded, and convex (i.e. if u and v are points in W, then $[u,v]'$ is a subset of W).

Lemma b. If $r = \frac{A+B}{2}$, $s = \frac{B+C}{2}$, and $t = \frac{C+A}{2}$, then there is a path P in W such that

(1) $P = [A;r;t;A]$ or $P = [r;B;s;r]$ or $P = [s;C;t;s]$ or $P = [t;r;s;t]$;
(2) If x is in K' and y is in P', then $|x - y| \leq l(P)$;
(3) $\left|\int_K f\right| \leq 4\left|\int_P f\right|$, moreover, $l(P) = \frac{l(K)}{2}$.

Lemma c. There is a sequence T such that $T_1 = K$ and for each positive integer n

(1) T_{n+1} is a triangular path in W and $l(T_{n+1}) = \frac{l(K)}{2^n}$;
(2) If x is in T_n' and y is in T_{n+1}', then $|x - y| \leq l(T_{n+1})$;
(3) $\left|\int_K f\right| \leq 4^n \left|\int_{T_{n+1}} f\right|$

Apr. 28

Prob. 39. Suppose Q is an open connected set and f is analytic on Q. If K is a closed polygonal path in Q and there is a point v such that if p is in K', then $[v;p]'$ is a subset of Q, then $\int_K f = 0$.

Prob. 40. If Q is an open connected set and z is a point in Q and f is a continuous function on Q that is analytic at each point of Q distinct from z, then the following statements are true:

1. If A, B, C are points such that if p is in $[B;C]'$, then $[A;p]'$ is a subset of Q, and $K = [A;B;C]$, then $\int_K f = 0$.
2. If K is a closed polygonal path in Q and there is a point v such that if p is in K', then $[v;p]'$ is a subset of Q, then $\int_K f = 0$.

Prob. 41. Suppose z is a point in the open connected set Q and the point function f is continuous on Q and analytic at each point of Q distinct from z. If K is a closed path in Q and there is a point v such that if p is in K' then $[v;p]'$ lies in Q, then $\int_K f = 0$.

Lemma a. If p is in K', then there is a positive number r such that if S is a point and $|S - p| < r$, then $[v;S]'$ lies in Q.

Lemma b. There is a positive number r such that if p is in K', S is a point, and $|S - p| < r$, then $[v;S]'$ lies in Q.

Prob. 42. Suppose K is a path, the point function h is continuous on K' and f is a sequence such that for each nonnegative integer n, f_n is the point function to which (z, w) belongs only in case z is a point not in K' and

$$w = n! \int_K \frac{h}{(I - z)^{n+1}}.$$

If n is a nonnegative integer, then $f'_n = f_{n+1}$.

Prob. 43. Suppose the point function f is analytic on the open connected set Q, K is a closed path in Q and there is a point v such that if p is in K', then $[v;p]'$ is a subset of Q. There is a point in Q not in K' and if z is such a point, then

$$\int_K \frac{f}{I - z} = f(z) \int_K \frac{1}{I - z}.$$

Prob. 44. Suppose K is a closed path and g is a point function to which the ordered pair (z, w) belongs only in case z is a point not in K' and

$$w = \int_K \frac{1}{I - z},$$

then

1. The function g is constant on each component of the complement of K'.
2. If z is a point in the unbounded complement of K' then $g(z) = 0$.

Prob. 45. If f is analytic on the open connected set Q, then f' is analytic on Q.

Prob. 46. If the entire function f has bounded final set, then f is constant.

Prob. 47. Suppose A is a point function from the set of nonnegative integers and n is a positive integer such that $A_n \neq 0$, and

$$F = A_0 + \sum_{t=1}^{n} A_t I^t \quad \text{polynomial of degree } n\ ,$$

then the following statements are true:

(1) F is an entire function.
(2) If $|z| > 1 + \sum_{t=0}^{n-1} \left|\frac{A_t}{A_n}\right|$, then $|F(z)| > |A_n||z|^{n-1}$.
(3) There is a point z such that $f(z) = 0$.

Def. 38. If f is a point function, $f^{(0)} = f$ and if k is a nonnegative integer, $f^{(k+1)} = (f^{(k)})'$.

Prob. 48. Suppose f is a point function that is analytic on the open connected set Q, w is in Q, and $r > 0$ such that if $|z - w| \leq 2r$, then z is in Q, and D is the component of the complement of $S_r(w)'$ to which w belongs, then

(1) If either $|\operatorname{Real}(z-w)| > r$ or $|\operatorname{Imag}(z-w)| > r$, then z is in the unbounded complement of $S_r(w)'$.
(2) D is the subset of Q to which z belongs only in case $|\operatorname{Real}(z-w)| < r$ and $|\operatorname{Imag}(z-w)| < r$.
(3) If p is in $S_r(w)'$, then $[w;p]'$ lies in Q.
(4) If t is a nonnegative integer and z lies in D, then

$$f^{(t)} = \frac{t!}{2\pi i} \int_{S_r(w)} \frac{f}{(I-z)^{t+1}}.$$

(5) If n is a positive integer and z is in D then

$$f(z) = \sum_{t=0}^{n-1} f^{(t)}(w) \frac{(z-w)^t}{t!} + \frac{1}{2\pi i} \int_{S_r(w)} \left(\frac{z-w}{I-w}\right)^n \frac{f}{I-z}.$$

(6) If $0 < s < r$ and $c > 0$, then there is a positive integer N such that if n is a positive integer and $|z - w| \leq s$, then

$$\left| f(z) - \sum_{t=0}^{N+n} \frac{f^{(t)}(w)}{t!}(z-w)^t \right| < c.$$

Def. 39. The exponential function E is the entire function f such that $f' = f$ and $f(0) = 1$.

Def. 40. The statement that f is a power series means that there is a point w and point sequence H such that

$$f = A_0 + \sum_{t=1}^{\infty} A_t (I-w)^t,$$

i.e., such that f is a sequence, f_0 is the constant function A_0 with initial set the plane and if n is a positive integer, then f_n is the point function $A_0 + \sum_{t=1}^{n} A_t(I-w)^t$ and in this case, f is said to be a power series about w with coefficient sequence A.

Remark. If w is in the initial set of the analytic function f then by an earlier problem, there is a positive number s such that the power series

$$f(w) + \sum_{t=1}^{\infty} \frac{f^{(t)}(w)}{t!}(I-w)^t$$

converges uniformly on the point set to which z belongs only in case $|z - w| \leq s$.
Prob. 49. If f is an analytic function and there is a positive number M such that if t is a nonnegative integer

$$|f^{(t)}(0)| < M,$$

then there is only one entire function of which f is a contraction and

(1) E is the power series $1 + \sum_{t=0}^{\infty} \frac{1}{t!} I^t$.

(2) If z is a point, then

$$E(z^\star) = E(z)^\star$$

and

$$E(z) = 1 + \int_{[0;z]} E.$$

(3) If each of z and w is a point, $E(z+w) = E(z)E(w)$.

(4) The contraction of E to the set of all real numbers is an increasing function with final set the set of all positive numbers.

Def. 41. The principal logarithmic function L is the function to which (z, w) belongs only in case z is a point such that either $\mathrm{Real}(z) > 0$ or $\mathrm{Imag}(z) \neq 0$ and

$$w = \int_{[1;z]} \frac{1}{I}.$$

Prob. 50. If K is a path from 1 to z in the initial set of L, then

(1) $L(z) = \int_K \frac{1}{I}$.

(2) L is analytic and L' is a proper contraction of $\frac{1}{I}$.

(3) If z is a point in the initial set of L and r is a positive number, then $L(r) + L(z) = L(rz)$.

Prob. 51. If z is a number, then $L(E(z)) = z$ and if z is a positive number, $E(L(z)) = z$, moreover if z is in the initial set of L, then $E(L(z)) = z$ and finally $E(-i\pi) = -1$ and $E(i\pi) = -1$.

Prob. 52. If z is a point and m is an integer, then $E(z + 2m\pi i) = E(z)$ and if z is a point, then $|E(z)| = E(\mathrm{Real}(z))$.

Remark. Analytic functions are as delicate as soap bubbles: if you alter the function at a single point, the function ceases to be analytic. More precisely, the value of the function on a sequence with a sequential limit point in the interior of the set of analyticity of the function determines the function everywhere in that set!