# NOTES FROM GEOMETRIC CONSTRUCTIONS IN HILBERT SPACES (MATH 6?) SUMMER 2005

HTTP://WWW.TANGENTSPACE.NET/MATHNOTES/HILBERTSPACEGEOM.PDF

**Def. 1.** $S = a_1, a_2, \ldots$ is a real number sequence. The statement that $S$ converges means

i) There is a number $T$.
ii) If $c$ is a positive number, then there is an integer $N$ such that if $n$ is an integer greater than $N$, then $|a_n - T| < c$.

**Notation.** We may say that $S$ converges to $T$ and we call, for each $i$, $a_i$ a term of the sequence.
**Def. 2.** $S = a_1, a_2, \ldots$ is a real number sequence and $b_1 = a_1$ and for each positive integer $n$, $b_{n+1} = b_n + a_{n+1}$. The statement that $S$ is summable means the number sequence $b_1, b_2, \ldots$ converges.

**Notation.** If $S$ is summable, then denote by $\sum a_n$ the number to which the sequence $b_1, b_2, \ldots$ converges.
**Prob. 1.** If $S$ is a summable number sequence then $S$ converges to 0.

**Def. 3.** $S$ is a real number sequence $b_1 = a_1^2$, $b_{n+1} = b_n + a_{n+1}^2$ for each positive integer $n$. The statement that $S$ is square summable means the number sequence $b_1, b_2, \ldots$ converges.

**Prob. 2.** If a number sequence $S$ is square summable, is $S$ summable?

**Def. 4.** $S$ is a real number sequence and $d$ is a number. $d \times S$ is the number sequence $da_1, da_2, \ldots$

**Prob. 3.** If $S$ is summable and $d$ is a number, then $d \times S$ is summable, moreover if $S$ if square summable, then $d \times S$ is square summable.

**Def. 5.** If each of $S = a_1, a_2, \ldots$ and $T = b_1, b_2, \ldots$ is a square summable number sequence, then $S + T$ is the number sequence $a_1 + b_1, a_2 + b_2, \ldots$.

**Prob. 4.** If each of $S$ and $T$ is a summable sequence, is $S + T$ summable?

**Prob. 5.** If each of $S = a_1, a_2, \ldots$ and $T = b_1, b_2, \ldots$ is a square summable number sequence, then the sequence $a_1b_1, a_2b_2, \ldots$ is summable.

**Notation.** If each of $S$ and $T$ is a square summable sequence, then denote by $Q(S, T)$ the number $\sum_{i=1}^{\infty} a_n b_n$, and note that $Q(S, T) = Q(T, S)$.
**Prob. 6.** If each of $S$ and $T$ is a square summable number sequence, then is $S + T$ square summable?

**Notation.** If $S$ is a number sequence and $d$ is a number, we shall denote $d \times S$ by $dS$. If $T$ is a number sequence, $S + T$ will be denoted by $S + T$. Finally, the number sequence each term of which is 0 will be denoted by $\emptyset$.
**Prob. 7.** Suppose $M$ is a number collection that contains the number 1, and if $n$ is a number in $M$ then $n + 1$ is in $M$. Does $M$ contain every positive integer?

**Def. 6.** The statement that a point collection $M$ is infinite means if $n$ is a positive integer, then $M$ has $n$ points.

**Def. 7.** $X$ is the set to which $x$ belongs only in case $x$ is a square summable number sequence.

We know that if each of $A$ and $B$ is in $X$ and $d$ is a number, then $dA+B$ is in $X$. Therefore $\{X,\mathbb{R},\times,+\}$ is a real linear space.
**Prob. 8.** If each of $A$, $B$, and $C$ is in $X$ and $d$ is a number, then
1) $Q(dA+B,C)=dQ(A,C)+Q(B,C)$
2) $Q(A,A)\geq 0$
3) $Q(B,B)=0$ only in case $B=\emptyset$
and if $A\neq\emptyset$ and $B\neq\emptyset$, then

$$\frac{[Q(A,B)]^2}{Q(A,A)Q(B,B)}\leq 1.$$

**Def. 8.** If $A$ is in $X$, $\|A\|=[Q(A,A)]^{\frac{1}{2}}$.

**Prob. 9.** If each of $A$ and $B$ is in $X$ and $d$ is a number, then
1) $\|A+B\|\leq\|A\|+\|B\|$
2) $\|dA\|=|d|\|A\|$
Note that $\|A\|=0$ only in case $A=\emptyset$ and $\|A\|\geq 0$.

**Def. 9.** $H$ is the real linear space $\{X,\mathbb{R},\times,+,Q\}$ and $Q$ is called an inner product. $\|\cdot\|=\|\cdot\|_Q$ is called the norm induced by the inner product $Q$. I.e., $H$ is a normed linear space.

**Prob. 10.** There is a sequence $S$, each term of which is in $X$, such that if $P$ is in $X$ and $c$ is a positive number, then there is a term $a$ in $S$ such that

$$\|P-a\|<c.$$

**Def. 10.** Suppose each of $A$ and $B$ is in $X$. The statement that $A$ is orthogonal to $B$ means $Q(A,B)=0$.

**Def. 11.** Suppose $L$ is a linear space. The statement that $M$ is a linear subspace of $L$ means if each of $x$ and $y$ is in $M$ and $c$ is a number, then $cx+y$ is in $M$.

**Def. 12.** The statement that $M$ is a dense subset of $H$ means $M$ is a subset of $X$ and if $P$ is in $X$ and $c$ is a positive number, then there is a point $a$ in $M$ such that

$$\|a-P\|<c.$$

**Prob. 11.** Is there a dense linear subspace of $H$ that is not $H$?

**Def. 13.** Suppose $x_1,x_2,\ldots$ is a point sequence in $H$. The statement that $x_1,x_2,\ldots$ converges means there is a point $y$ such that if $c>0$, then there is a positive integer $N$ such that if $n$ is an integer, $n>N$, then $\|x_n-y\|<c$.

**Prob. 12.** There is a point sequence $x_1,x_2,\ldots$ each term of which is in $H$ such that if $P$ is a point in $H$, then there is a number sequence $c_1,c_2,\ldots$ such that the sequence

$$c_1x_1, c_1x_1+c_2x_2, c_1x_1+c_2x_2+c_3x_3,\ldots$$

converges to $P$. Moreover, there is such a point sequence $x_1,x_2,\ldots$ such that $Q(x_i,x_j)=0$ if $i\neq j$.

**Prob. 13.** If $x_1,x_2,\ldots$ is a reversible point sequence in $H$, then there is a sequence $T_1,T_2,\ldots$ each term of which is a point sequence such that
1) for each positive integer $n$, each term in $T_n$ is a term of the sequence $x_1,x_2,\ldots$
2) if $x_i$ is the $m$th term of $T_l$ and $x_i$ is the $n$th term of $T_k$, then $l=k$ and $m=n$.

**Def. 14.** Suppose $X$ is a real linear space. The statement that $f$ is a functional on $X$ means $f$ is a function from $X$ into the real numbers.
The statement that $f$ is a linear functional on $X$ means $f$ is a functional on $X$ and if each of $x$ and $y$ is in $X$ and $a$ is a number, then

$$f(ax + y) = af(x) + f(y).$$

Finally, if $X$ is a normed linear space with norm $\|\cdot\|$, then the statement that $f$ is a bounded linear functional on $X$ means $f$ is a linear functional on $X$ and there is a positive number $B$ such that

$$|f(x)| \leq B\|x\| \text{ for all } x \in X.$$

**Prob. 14.** If $f$ is a bounded linear functional on $H$ then there is a point $P$ in $H$ such that

$$f(x) = Q(x, P)$$

for all $x$ in $H$.

**Def. 15.** Suppose $M$ is a subset of $H$. The statement that $P$ is a limit point of $M$ means if $c > 0$, then there is a point $x$ in $M$ distinct from $P$, such that

$$\|P - x\| < c.$$

**Def. 16.** Suppose $M$ is a subset of $H$. The statement that $M$ is closed means if $P$ is a limit point of $M$, then $P$ is a point in $M$.

**Prob. 15.** Suppose $x_1, x_2, \ldots$ is a point sequence in $H$ such that if $c$ is a positive number, then there is a positive number $N$ such that if $m$ and $n$ are integers greater than $N$, then

$$\|x_m - x_n\| < c.$$

Does the sequence $x_1, x_2, \ldots$ converge?

**Def. 17.** The statement that $I$ is an interval in $H$ means there are points $A$ and $B$, and $I$ is the set to which $x$ belongs only in case $x = (1 - t)A + tB$ for some number $t$ in $[0, 1]$.

**Def. 18.** The statement that $\alpha$ is an arc in $H$ means there is a function $h$ from $[0, 1]$ into $H$ such that
1) $h$ is continuous
2) $h$ has an inverse called $h^{-1}$
3) $h^{-1}$ is continuous

and $\alpha$ is $h([0, 1])$, i.e. $\alpha$ is the final set (range) of $h$. $h(0)$ and $h(1)$ are called endpoints of $\alpha$.

**Def. 19.** Suppose $M$ is a subset of the numbers and $f$ is a function from $M$ into $H$. The statement that $f$ is continuous means if $(x, y)$ is in $f$ and $c$ is a positive number, then there is a positive number $d$ such that if $z$ is in $M$ and $|z - x| < d$, then $\|f(z) - y\| < c$.

**Def. 20.** Suppose $W$ is a subset of $H$ and $g$ is a function from $W$ into the numbers. The statement that $g$ is continuous means if $(u, v)$ is in $g$ and $c$ is a positive number, then there is a positive number $d$ such that if $z$ is in $W$ and $\|z - u\| < d$, then $|g(z) - v| < c$.

**Prob. A.** Find an arc $\alpha$ in the linear subspace spanned by $\phi_1$ and $\phi_2$ such that if $\alpha$ is $h([0, 1])$, $h(0) = \phi$, $h(1) = \phi_1$, and there is only one point $P$ in $\alpha$ such that $Q(P - h(0), P - h(1)) = 0$. Moreover, if $P = h(t)$, $Q(h(t)-h(\delta), h(u)-h(t)) = 0$ where $\delta < t < u$, and finally if $u < v < w < x \leq t$, $Q(h(u)-h(v), h(w)-h(x)) \neq 0$.